ST. GREGORIOS COLLEGE KOTTARAKARA

Student’s Project- UG

2023-2024

BIOSYNTHESIS OF SILVER NANOPARTICLES

PROJECTWORK

Submitted to the University of Kerala in partial fulfillment of the requirements for the award of the Degree of Bachelor of Science in Physics

BY

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AN INVENTORY ON MEDICINAL PLANTS OF VALATHUNGAL SACRED GROVES, KOLLAM

Dissertation work

Submitted to the University of Kerala in partial fulfilment of the requirement for the award of BSc Degree course 2021- 2024

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ADSORPTION OF Ni^{2+} ON AMINO PROPYL - FUNCTIONALIZED ATTAPULGITE

DISSERTATION

SUBMITTED TO THE UNIVERSITY OF KERALA
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Diversity of Dragonflies (Insecta: Odonata) of Kottarakara region, Kerala

Dissertation submitted to the University of Kerala in partial fulfillment of requirements for the Degree of

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Dissertation submitted to the University of Kerala to partially fulfill the requirements for the degree of Bachelor of Science in Zoology.

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DECENTRALISATION AND SOCIAL JUSTICE IN KERALA AN INVESTIGATION INTO THE EFFICIENCY OF SOCIAL SECURITY MEASURES IMPLEMENTED THROUGH LSGs

A dissertation submitted to the University of Kerala in partial fulfillment of the requirement for the award of Bachelor of Arts Degree in Political Science

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AN INTRODUCTION TO CRYPTOGRAPHY

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A STUDY ON THE PREVALENCE OF ONLINE FRAUDS AND SCAMS IN KOTTARAKARA MUNICIPALITY

Project submitted to

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FIRST DEGREE PROGRAMME IN MATHEMATICS
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A STUDY ON INTEGRAL EQUATIONS

A STUDY ON INTEGRAL EQUATIONS

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Contents

INTRODUCTION

Historically, Fourier (1768-1830) is the intiator of the theory of integral equations. Du Bois Reymond first suggest the term integral equation in 1888 Pioneering systematic areas research goes back to the work of Volterra, Fredholm and Hilbert in the late 19th and early 20th centuries. Various physical problems in physics and other applied fields culminate into intial value problems or boundary value problems. Although it is equivalent to frame the problem in the form of (ordinary and partial) differential equation or in the form of integral equations, but it is prefered to choose the integral form due to two main reasons. Firstly the solution of integral equation is much easier than the original boundary values or the intial value problems. The second reason lies in the fact that integral equations are better suited to approximate methods than differential equations. Moreover, integral equations develop as representation formulae for the solution of differential equations.

CHAPTER 1

PRELIMINARIES

1.1 Definition

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation is:

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)\,dt$$

where $k(x,t)$ is the kernel of the integral equation, $\alpha(x)$ and $\beta(x)$ are the limits of integration.

1.1.1 Definition

A linear integral equation is an equation involving an unknown function $y(x)$ which appears under an integral sign and is linear in $y(x)$ and its derivatives. It typically takes the form:

$$\int_a^b K(x,t)y(t)dt = f(x)$$

where $K(x,t)$ and $f(x)$ are known functions and the goal is to find the unknown function $y(x)$. An example of a linear integral equation is the Fredholm integral equation of the second kind:

$$\int_0^1 K(x,t)y(t)dt = f(x)$$

where $K(x,t)$ and $f(x)$ are given functions.

A nonlinear integral equation is an equation where an unknown function appears under an integral sign, and the equation itself involves nonlinear operations on that function.

1.2 Classification of Linear Integral Equations

The two main classes are namely Fredholm and Volterra integral equations and the 4 related types:

- Fredholm Integral equation
- Volterra integral equation
- Integro-differential equation
- Singular integral equation
- Volterra-Fredholm integral equation
- Volterra -Fredholm integral-differential equation

1.2.1 Fredholm Linear Integral Equation

The standard form of Fredholm linear integral equation, where the limits of integration a and b, are constants are given by:
$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \quad a \leq x,t \leq b \quad (1)$ Where λ is a parameter and $k(x,t)$ is the kernel.

- When $\phi(x) = 0$, equation (1) becomes $0 = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \quad (2)$
- When $\phi(x) = 1$,equation(1) becomes $u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \quad (3)$

where equation (2) is called the Fredholm integral equation of the first kind, equation(3) is called the Fredholm integral equation of the second kind.

1.2.2 Volterra Linear Integral Equation

The standard form of Volterra Linear integral equation, where the limits of integration are functions rather than constants:

$$\phi(x)u(x) = f(x) + \lambda \int_a^x k(x,t) \cdot u(t)\,dt \quad (4)$$

When $\phi(x) = 0$, equation (4) becomes:

$$f(x) + \lambda \int_a^x k(x,t) \cdot u(t)\,dt = 0 \quad (5)$$

This integral equation is called Volterra integral equation of the first kind. When $\phi(x) = 1$, equation (4) yields:

$$u(x) = f(x) + \lambda \int_a^x k(x,t) \cdot u(t)\, dt$$

This integral equation is called Volterra integral equation of the second kind. **Note**: If $f(x) = 0$, then the resulting integral equation is called a homogeneous integral equation; otherwise, it is called a non-homogeneous integral equation

Question 1: Classify the integral equation

$$u(x) = \frac{1}{2} + x - \int_0^1 (x-1) \cdot u^2(t)\, dt$$

as Fredholm or Volterra integral equation, homogeneous or non-homogeneous.

Solution:

- It is non-homogeneous since $f(x) = \frac{1}{2} + x \neq 0$.
- The limits of integration are constant.
- The function $u(x)$ appears twice.
- The unknown function appears under the integral sign.

Question 2: Classify the following equation

$$u(x) = 1 - 2x^2 + \int_0^t f(t)\, dt, \quad u(0) = 0$$

as Fredholm or Volterra integro-differential equation and homogenous or non homogenous.

Solution:

- The equation includes both differential and integral operators.
- The upper limit of the integral is a variable.

Thus, it is a Volterra integro-differential equation.

1.3 Solution of an Integral Equation

A solution of an integral equation on the interval of integration is a function $u(x)$ such that it satisfies the given equation. That is, if the given solution is substituted on the right-hand side of the equation, the output of this direct substitution must yield the left-hand side.

Question 1: Show that $u(x) = e^x$ is a solution of the equation

$$u(x) = 1 + \int_0^x u(t)\, dt$$

Solution:

$$\begin{aligned} \text{Put } u(x) &= e^x \\ \text{RHS} &= 1 + \int_0^x e^t\, dt \\ &= 1 + \left[e^t\right]_0^x \\ &= 1 + (e^x - e^0) \\ &= 1 + e^x - 1 \\ &= e^x \\ &= \text{LHS} \end{aligned}$$

Question:2 Show that $u(x) = x$ is a solution of the following Fredholm integral equation:

$$u(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)u(t)\, dt$$

Solution:

$$\begin{aligned}
\text{Put } u(x) &= x \\
\text{RHS} &= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)t\,dt \\
&= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (xt+t^2)\,dt \\
&= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{1}{2}(x\cdot[1-0]) + \frac{1}{3}([1-0])\right] \\
&= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left(\frac{x}{2} + \frac{1}{3}\right) \\
&= \frac{5x}{6} - \frac{1}{9} + \frac{x}{6} + \frac{1}{9} \\
&= \frac{5x}{6} + \frac{x}{6} \\
&= \frac{6x}{6} \\
&= x \\
&= u(x)
\end{aligned}$$

1.3.1 Leibnitz Rule

The Leibniz rule for differentiation under the integral sign is given by:

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x,t)\,dt\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}\partial(t) + F(x,\beta(x))\frac{d\beta(x)}{dx} - F(x,\alpha(x))\frac{d\alpha(x)}{dx}$$

1.3.2 Lemma

If n is a positive integer, then:

$$\int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)\, dx_n\, dx_{n-1} \ldots\, dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt$$

CHAPTER-2

FREDHOLM INTEGRAL EQUATIONS

2.1 Converting BVP to Fredholm Equation

Consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1 \qquad (2)$$

with boundary conditions:

$$y(0) = \alpha, \quad y(1) = \beta \qquad (3)$$

Set$y''(x) = u(x) \qquad (4)$
and integrate(4) both sides from 0 to x:

$$\int_0^x y''(t)dt = \int_0^x u(t)dt \qquad (5)$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt \qquad (6)$$

where the intial condition $y'(0)$ is not given , so it will be determined later using the boundary condition at $x = 1$. Integrating on the both sides from

0 to x

imlpies

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dt \qquad (7)$$

or equivalently:

$$y(x) = \alpha + xy'(0) + \int_0^x (x-t)u(t)dt \qquad (8)$$

To determine $y'(0)$, we substitute $x = 1$ into both sides of (8) and using the boundary condition at $y(1) = \beta, we find$:

$$y(1) = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt \qquad (9)$$

which gives:

$$\beta = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt \qquad (10)$$

Rearranging gives:

$$y'(0) = (\beta - \alpha) - \int_0^1 (1-t)u(t)dt \qquad (11)$$

Substituting (11) in (8):

$$y(x) = \alpha + (\beta - \alpha)x - \int_0^1 (1-t)u(t)dt + \int_0^x (x-t)u(t)dt \qquad (12)$$

Substituting (12) and (4) in equation (2) yields

$$u(x)+\alpha g(x)+(\beta-\alpha)xg(x)-\int_0^1 xg(x)(1-t)u(t)dt+\int_0^x g(x)(x-t)u(t)dt = h(x) \qquad (13)$$

From (13) we get:

$$u(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x) - g(x)\int_0^x (x-t)u(t)dt$$
$$+ xg(x)\left[\left(\int_0^x (1-t) + \int_x^1 (1-t)\right) u(t)dt\right] \quad (14)$$

which gives:

$$u(x) = f(x) + \int_0^x t(1-x)g(x)u(t)dt + \int_x^1 x(1-t)g(x)u(t)dt \qquad (15)$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt \qquad (16)$$

where:

$$f(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x) \qquad (17)$$

The kernel $K(x,t)$ is given by:

$$K(x,t) = \begin{cases} t(1-x)g(x), \text{for} & 0 \leq t \leq x \\ x(1-t)g(x), \text{for} & x \leq t \leq x \end{cases}$$

Question 1 : Derive an equivalent Fredholm integral equation to the following boundary value problem:

$$y''(x) + y(x) = x, \quad 0 < x < \pi \qquad (18)$$

subject to the boundary conditions:

$$y(0) = 1, \quad y(\pi) = \pi - 1 \qquad (19)$$

Solution

Let $y''(x) = u(x) \qquad (20)$.

Integrating both sides of (20) from 0 to x:

$$\int_0^x y''(t)dt = \int_0^x u(t)dt \qquad (21)$$

or equivalently:

$$y'(x) = y'(0) + \int_0^x u(t)dt \qquad (22)$$

Note that $y'(0)$ can be determined later by using the boundary conditions $x = \pi$

Integrating (22) yields:

$$y(x) = 1 + xy'(0) + \int_0^x (x - t)u(t)dt \qquad (23)$$

put$x = \pi$ $y(\pi) = 1 + xy'(0) + \int_0^\pi (\pi - t)u(t)dt \qquad (24)$

and solving for $y'(0)$, we obtain:

$$y'(0) = \frac{1}{\pi}\left((\pi - 2) - \int_0^\pi (\pi - t)u(t)dt\right) \qquad (25)$$

Substituting (24) for y'(0) into (23) yields:

$$y(x) = 1 + \frac{x}{\pi}\left((\pi - 2) - \int_0^{\pi} (x-t)u(t)dt\right) + \int_0^{x} (x-t)u(t)dt \qquad (26)$$

Substituting (26) and (25) these expressions into equation (18) gives:

$$u(x) = x-1-\frac{x}{\pi}(\pi-2)-\frac{x}{\pi}\int_0^{\pi}(x-t)u(t)dt-\int_0^{x}(x-t)u(t)dt-\frac{x}{\pi}\int_0^{x}(x-t)u(t)dt \qquad (27)$$

$$u(x) = \frac{2x-\pi}{\pi} - \int_0^{x} \frac{t}{\pi}(x-\pi)u(t)dt - \int_x^{\pi} \frac{x(t-\pi)}{\pi}u(t)dt$$

where ,

$$K(x,t) = \begin{cases} \frac{t(x-\pi)}{\pi}, 0 \le t \le x \\ \frac{x(t-\pi)}{\pi}, x \le t \le \pi \end{cases}$$

Question 2: Convert the Fredholm integral equation

$$u(x) = \lambda \int_0^1 K(x,t)u(t)dt$$

$$K(x,t) = \begin{cases} x(t-x) 0 \le x \le 1, \\ t(1-x) t \le x \le 1 \end{cases} \qquad (1)$$

into the boundary value problem

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u(1) = 0$$

Solution:

We have

$$u(x) = \lambda \left[\int_0^x t(1-x)u(t)dt + \int_x^1 x(1-t)u(t)dt \right] \qquad (1)$$

Differentiating with respect to x and using the Leibniz formula, we get

$$\frac{d}{dx}u(x) = \lambda - \int_0^x tu(t)dt + x(1-x)u(x) + \int_x^1 (1-t)u(t)dt$$

$$-\lambda x(1-x)u(x)$$

$$\frac{du}{dx} = \lambda \left[\int_0^x -tu(t)dt + \int_x^1 (1-t)u(t)dt \right]$$

Differentiating again

$$\frac{d^2u}{dx^2} = \lambda \int_0^x 0 \times -tu(t)dt + \lambda(-x)u(x) + \int_x^1 0 \times (1-t)u(t)dt - \lambda(1-\lambda)u(x) - \lambda u(x) = -\lambda u(x)$$

$$\frac{d^2u}{dx^2} + \lambda u(x) = 0$$

Therefore $u"(x) + \lambda u(x) = 0$

From (1), we have $u(0) = 0 = u(1)$

2.2 Solution of Fredholm Integral Equation

Consider a second kind Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (1)$$

We define an integral operator

$$K[\phi(x)] = \int_a^b K(x,t)\phi(t)dt$$

$$K^2[\phi(x)] = K[K(\phi(x))]$$

There (1) can be written as

$$u(x) = f(x) + \lambda K[u(x)]$$

2.2.1 Theorem

The Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (1)$$

is such that

- $K(x,t)$ is a non-zero real valued continuous function in the rectangle $R = I \times I$, where $I = [a,b]$ and $|K(x,t)| < M$ in R.

- $f(x)$ is a non-zero real valued and continuous function on I.

- λ is a constant satisfying the inequality $|\lambda| < \frac{1}{M(b-a)}$.

Then (1) has a solution, and only one continuous function on the interval I, and this is given by the absolutely and uniformly convergent series:

$$u(x) = f(x) + \lambda K[f(x)] + \lambda^2 K^2[f(x)] + \cdots$$

2.3 Resolvent Kernel for Fredholm Integral Equation

Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (1)$$

The iterated kernels are defined by

$$K_1(x,t) = K(x,t)$$

$$K_{n+1}(x,t) = \int_a^b K(x,y)K_n(y,t)dy, \quad n = 1, 2, 3, \cdots$$

and the solution of (1) is given by

$$u(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)f(t)dt$$

where

$$R(x,t;\lambda) = K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \cdots = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$$

2.3.1 Neumann Series

The infinite series

$$K_1 + \lambda K_2 + \lambda^2 K_3 + \cdots$$

is called the Neumann Series.

2.3.2 Resolvent Kernel

The function $R(x,t;\lambda)$ is called the Resolvent Kernel.

Question 1: Obtain the resolvent kernel associated with the kernel

$$K(x,t) = 1 - 3xt$$

in the interval $[0,1]$ and solve the integral equation

$$u(x) = 1 + \lambda \int_0^1 (1 - 3xt)u(t)dt$$

.

Solution:

We have $K(x,t) = 1 - 3xt$. We know that the iterated kernels are given by the relation:

$$K_1(x,t) = K(x,t)$$

$$K_{n+1}(x,t) = \int_a^b K(x,y)K_n(y,t)dy$$

Therefore,

$$K_1(x,t) = 1 - 3xt$$

$$K_2(x,t) = \int_0^1 K(x,y)K_1(y,t)dy = \int_0^1 (1-3xy)(1-3ty)dy = \int_0^1 (1-3ty-3xy+9xyt)dy$$

$$= \int_0^1 (1 - 3ty - 3xy + 9xyt^2)dy = \left[y - \frac{3t}{2}y^2 - \frac{3x}{2}y^2 + \frac{9xt}{3}y^3\right]_0^1$$

$$= 1 - \frac{3t}{2} - \frac{3x}{2} + 3xt$$

$$K_3(x,t) = \int_0^1 K(x,y)K_2(y,t)dy$$

$$K_3(x,t) = \int_0^1 (1 - 3xy)(1 - \frac{3}{2}y - \frac{3}{2}t + 3yt)dy = \frac{1}{4}(1 - 3xt)$$

$$K_4(x,t) = \int_0^1 K(x,y)K_3(y,t)dy = \frac{1}{4}\int_0^1 (1-3xy)(1-3yt)dy = \frac{1}{4}\left[1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right]$$

The Resolvent Kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \lambda^4 K_4 + \cdots$$

$$= (1-3xt)+\lambda\left(\frac{1-3t}{2} - \frac{3x}{2} + 3xt\right)+\frac{\lambda^2}{4}(1-3xt)+\frac{\lambda^4}{4}\left[1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right]+\cdots$$

$$= (1 - 3xt)\left(1 + \frac{\lambda^2}{4}\right) + \lambda\left(1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right)\left(1 + \frac{\lambda^2}{4}\right) + \cdots$$

$$R(x,t;\lambda) = \left(1 + \frac{\lambda^2}{4} + \cdots\right)\left[(1-3xt) + \lambda\left(1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right)\right]$$

$$= \frac{1}{1-\frac{\lambda^2}{4}}\left[(1-3xt) + \lambda\left(1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right)\right]$$

$$= \frac{4}{4-\lambda^2}\left[(1-3xt) + \lambda\left(1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right)\right]$$

The solution of the integral equation is given by

$$u(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t) dt$$

where $K(x,t) = 1 - 3xt$, then

$$R(x,t;\lambda) = \frac{4}{4-\lambda^2}\left[(1-3xt) + \lambda\left(1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right)\right]$$

Thus, the solution of the given integral equation is

$$u(x) = 1 + \frac{4\lambda}{4-\lambda^2}\int_0^2 \left[(1-3xt) + \lambda\left(1 - \frac{3t}{2} - \frac{3x}{2} + 3xt\right)\right] dt$$

$$u(x) = 1 + \frac{4\lambda}{4-\lambda^2}\left[t - \frac{3xt^2}{2} + \lambda\left(t - \frac{3t^2}{4} - \frac{3xt}{2} + \frac{3xt^2}{2}\right)\right]_0^1$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left[1 - \frac{3x}{2} + \lambda\left(1 - \frac{3}{4} - \frac{3x}{2} + \frac{3}{2}\right)\right]$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left[1 - \frac{3x}{2} + \frac{\lambda}{4}\right]$$

$$= \frac{4 + 4\lambda - 6x\lambda}{4-\lambda^2}$$

, $\lambda \neq 2$

2.4 The Adomian Decomposition Method

The method provides the solution in a series. In the decomposition method, we express the solution $u(x)$ of the integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, a \leq x \leq b \qquad (1)$$

in the form defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2)$$

Substituting the decomposition into both sides, we get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \qquad (3)$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots = f(x)+\lambda \int_a^b K(x,t)u_0(t)dt+\lambda \int_a^b K(x,t)u_1(t)dt+\lambda \int_a^b K(x,t)u_2(t)dt$$

The components $u_0(x), u_1(x), u_2(x), \cdots$ of the unknown function $u(x)$ are completely determined in a recurrent manner if we set:

$$u_0(x) = f(x) \qquad (5)$$

$$u_1(x) = \lambda \int_a^b K(x,t)u_0(t)dt \qquad (6)$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt \qquad (7)$$

and so on. (1) can be written in a recursive manner as:

$$u_0(x) = f(x) \qquad (8)$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt, \quad n \geq 0 \qquad (9)$$

With these components determined, the solution $u(x)$ can be readily determined in a series form.

Question1: Consider the Fredholm integral equation of the second kind

$$u(x) = \frac{9}{10}x^2 + \int_0^1 \frac{1}{2}x^2t^2u(t)dt$$

Solution:

$f(x) = \frac{9}{10}x^2, \lambda = 1$ and $K(x,t) = \frac{1}{2}x^2t^2$. Next, we have to evaluate $u_0(x), u_1(x), u_2(x), \cdots$. The series solution is:

$$u_0(x) = \frac{9}{10}x^2$$

$$u_1(x) = \int_0^1 \frac{1}{2}x^2t^2u_0(t)dt = \int_0^1 \frac{1}{2}x^2t^2\frac{9}{10}t^2dt = \frac{9}{100}x^2$$

$$u_2(x) = \int_0^1 \frac{1}{2}x^2t^2u_1(t)dt = \int_0^1 \frac{1}{2}x^2t^2\frac{9}{100}t^2dt = \frac{9}{1000}x^2$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots = \frac{9}{10}x^2 + \frac{9}{100}x^2 + \frac{9}{1000}x^2 + \cdots$$

The solution in a closed form is:

$$u(x) = x^2$$

Question 2: Consider the Fredholm integral equation

$$u(x) = e^x - 1 + \int_0^1 tu(t)dt$$

Solution:

$$u_0(x) = f(x) = e^x - 1$$

$$u_1(x) = \int_0^1 tu_0(t)dt = \int_0^1 t(e^t - 1)dt = \frac{1}{2}$$

$$u_2(x) = \int_0^1 tu_1(t)dt = \int_0^1 \frac{1}{2}tdt = \frac{1}{4}$$

$$u(x) = e^x - 1 + \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{4} + ...)$$

The solution is in the closed form:

$$u(x) = e^x$$

2.5 The Method of Successive Substitutions

This method introduces the solution of the integral equation in a series form through evaluating single integral and multiple integrals as well. w In this method, we set $x = t$ and $t = t_1$ in the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \quad a \leq x \leq b$$

to obtain

$$u(t) = f(t) + \lambda \int_a^b K(t,t_1)u(t_1)dt_1$$

Replacing $u(t)$ in the right-hand side , we obtain

$$u(x) = f(x) + \lambda \int_a^b K(x,t)f(t)dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)u(t_1)dt_1dt$$

Question 1: Consider the Fredholm integral equation

$$u(x) = x + \lambda \int_0^1 xtu(t)dt$$

Solution

The zeroth approximation may be selected by

$$u_0(x) = 0$$

substituting this in the right-hand side

$$u_1(x) = x$$

follows immediately. Proceeding in the same manner, we find that

$$u_2(x) = x + \lambda \int_0^1 xt^2 dt$$

so that

$$u_2(x) = x + \frac{\lambda}{3}x$$

In a similar manner, we obtain

$$u_3(x) = x + \lambda \int_0^1 xt\left(1 + \frac{\lambda}{3}t\right) dt$$

which yields

$$u_3(x) = x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x$$

Generally, we obtain for the nth approximation

$$u_n(x) = x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x + \cdots + \frac{\lambda^{n-1}}{3^{n-1}}x, \quad n \geq 1$$

Consequently, the solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(x + \frac{\lambda}{3}x + \frac{\lambda^2}{9}x + \cdots\right) = \frac{1}{1-\lambda}x, \quad 0 < \lambda < 3$$

Using the new selection of $u_0(x)$ in the right-hand side the first approximation

$$u_1(x) = x + \frac{\lambda}{3}x$$

is readily obtained. Proceeding as before, we thus obtain

$$u_2(x) = x + \lambda \int_0^1 xt \left(t + \frac{\lambda}{3} t \right) dt$$

which gives

$$u_2(x) = x + \frac{\lambda}{3} x + \frac{\lambda^2}{9} x$$

In a parallel manner, we find

$$u_n(x) = x + \frac{\lambda}{3} x + \frac{\lambda^2}{9} x + \cdots + \frac{\lambda^n}{3^n} x, \quad n \geq 1$$

2.6 The Direct Computation Method

We next introduce an efficient traditional method for solving Fredholm integral equations of the second kind ($K(x,t)$ expressed in the form defined by .

$$K(x,t) = g(x)h(t)$$

Accordingly, the equation (**??**) becomes

$$u(x) = f(x) + \lambda g(x) \int_a^b h(t)u(t)dt$$

It is clear that the definite integral at the right-hand side of (t. This means that the definite integral in the right-hand side of (α, where α is a constant.

In other words, we may write

$$\int_a^b h(t)u(t)dt = \alpha$$

It follows that equation becomes

$$u(x) = f(x) + \lambda\alpha g(x)$$

It is thus obvious that the solution $u(x)$ is completely determined by (α. This can be easily done by substituting Eq.

Question 1:

Consider here the Fredholm integral equation

$$u(x) = \sin^{-1}(x) + \left(\frac{\pi}{2} - 1\right)x - \int_0^1 xu(t)dt$$

Solution

Applying the modified decomposition method , we first split the function $f(x)$ into

$$f_0(x) = \sin^{-1}(x)$$

$$f_1(x) = \left(\frac{\pi}{2} - 1\right)x$$

Therefore, we set

$$u_0(x) = \sin^{-1}(x)$$

$$u_1(x) = \left(\frac{\pi}{2} - 1\right) x - x \int_0^1 \sin^{-1}(t)dt = 0$$

Consequently, the components $u_n(x) = 0$ for $n \geq 1$. The exact solution is readily obtained:

$$u(x) = \sin^{-1}(x)$$

Question 2 Consider the Fredholm integral equation

$$u(x) = \sin x + \cos x - 2x + \frac{\pi}{2} + \int_0^{\pi} (x - t)u(t)dt$$

Solution

We first split the function $f(x)$ into

$$f_0(x) = \sin x + \cos x$$

$$f_1(x) = -2x + \frac{\pi}{2}$$

We then set

$$u_0(x) = \sin x + \cos x$$

$$u_1(x) = -2x + \frac{\pi}{2} + \int_0^{\pi} (x - t)u_0(t)dt = 0$$

Consequently, the components $u_n(x) = 0$ for $n \geq 1$. The exact solution is readily obtained:

$$u(x) = \sin x + \cos x$$

CHAPTER-3

VOLTERRA INTEGRAL EQUATIONS

3.1 Converting IVP to Volterra equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \quad (1)$$

Subject to the initial conditions:

$$y(0) = \alpha, \quad y'(0) = \beta \quad (2)$$

Where a and β are constants. The functions $p(x)$ and $q(x)$ are analytic functions and $g(x)$ is continuous throughout the interval of discussion. To achieve our goal, set:

$$y''(x) = u(x) \quad (3)$$

where $u(x)$ is a continuous function.
integrating both sides of (3) from 0 to x yields

$$\int_0^x y''(x)\, dx = \int_0^x u(t)\, dt \quad (4)$$

y'(x) - y'(0) = $\int_0^x u(t)\, dt$

or equivalently

y'(x)=$\beta + \int_0^x u(t)dt \quad (5)$

integrating both sides of equation (5) from 0 to x

$$\text{y(x)-y(0)}=\beta(x) + \int_0^x \int_0^x u(t)dtdt \quad (6)$$

$$\text{y(x)} = \alpha + \beta x + \int_0^x (x-t)\, u(t)\, dt \quad (7)$$

substituting (3),(5) and (7) into the IVP

$$\text{u(x) + p(x)} \left[\beta + \int_0^x u(t)\, dt\right] + q(x)\left[\alpha + \beta x + \int_0^x (x-t)u(t)dt\right] = g(x) \quad (8)$$

(8) can be written in the standard volterra integral equation form

$$u(x) = f(x) - \int_0^x k(x,t)u(t)dt$$

where

$$u(x) = f(x) - \int_0^x k(x,t)u(t)dt$$

$$k(x,t) = p(x) + q(x)(x-t)$$

$$f(x) = g(x) - [\beta p(x) + aq(x) + \beta x q(x)]$$

Question 1: Transform the initial value equation $y''' - 3y'' - 6y' + 5y = 0$ subject to the initial conditions $y(0) = y'(0) = y''(0) = 1$, into an equivalent Volterra integral equation.

Solution:

Let $y'''(x) = u(x)$ (1) Integrating both sides of (1) from 0 to x and using the initial condition $y''(0) = 1$, we get

$$y''(x) = 1 + \int_0^x u(t)dt. \quad (2)$$

Integrating (2) twice and using the proper initial condition, we find

$$y'(x) = 1 + x + \int_0^x \int_0^t u(t)dtdt$$

.

and

$$y(x) = 1 + x + \frac{1}{2}x^2 + \int_0^x \int_0^x \int_0^x u(t)\, dt\, dt\, dt \quad (3)$$

$$y'(x) = 1 + x + \int_0^x (x-t)u(t)\, dt \quad (4)$$

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\, dt \quad (5)$$

Substitute (1), (2), (3), (4), (5) in the IVP:

$$u(x) = 4 + x - \frac{5}{2}x^2 + \int_0^x \left(3 + 6(x-t) - \frac{5}{2}(x-t)^2\right) u(t)\, dt$$

Question 2: Find the equivalent Volterra integral equation to the following initial value problem $y''(x) + y(x) = \cos(x)$, $y(0) = 0$, $y'(0) = 1$

Solution:

Set $y''(x) = u(x) \quad (1)$.

Integrating both sides from 0 to$x$

$$
\begin{aligned}
\int_0^x y''(x)\,dx &= \int_0^x u(t)\,dt \\
[y'(x)]_0^x &= \int_0^x u(t)\,dt \\
y'(x) - y'(0) &= \int_0^x u(t)\,dt \\
y'(x) - 1 &= \int_0^x u(t)\,dt \\
y'(x) &= 1 + \int_0^x u(t)\,dt \quad (2)
\end{aligned}
$$

integrating (2) from 0 to x

$$
\begin{aligned}
\int_0^x y'(x)\,dx &= \int_0^x \left(1 + \int_0^x u(t)\,dt\right)\,dt \\
[y(x)]_0^x &= \int_0^x 1\,dt + \int_0^x \int_0^x u(t)\,dt\,dt \\
y(x) - y(0) &= [t]_0^x + \int_0^x \int_0^x u(t)dtdt \\
y(x) &= [x - 0] + \int_0^x \int_0^x u(t)\,dt\,dt \\
y(x) &= x + \int_0^x (x-t)u(t)\,dt \quad (3)
\end{aligned}
$$

Substitute (1) and (3) in the IVP:

$$
\begin{aligned}
u(x) + x + \int_0^x (x-t)\cdot u(t)\,dt &= \cos(x) \\
u(x) &= \cos(x) - x - \int_0^x (x-t)u(t)\,dt
\end{aligned}
$$

The equivalent Volterra integral equation.

3.2 Converting Volterra Equation to an ODE

In this section, we present the technique that converts a Volterra integral equation of the second kind to an equivalent differential equation. This may be achieved by applying the important Leibniz rule for differentiating an integral equation.

Question1: Find $\frac{d}{dx}\int_0^x (x-t)^2 u(t)\,dt$.

Solution:

We know that

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x,t)dt\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}\partial t + F(x,\beta(x))\cdot\frac{d\beta(x)}{dx} - F(x,\alpha(x))\cdot\frac{d\alpha(x)}{dx}$$

Here, $\alpha(x) = 0$ and $\Rightarrow \alpha'(x) = 0$, $\beta(x) = x$ and $\Rightarrow \beta'(x) = 1$,
$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x-t)^2 u(t) = 2(x-t)u(t)$. Therefore,

$$\frac{d}{dx}\left(\int_0^x (x-t)^2 u(t)\,dt\right) = \int_0^x 2(x-t)u(t)\,dt$$

Question 2: Find $\frac{d}{dx}\int_0^x (x-t)u(t)\,dt$.

Solution:

$$\begin{aligned} \alpha(x) &= 0, \quad \beta(x) = x, \\ \alpha'(x) &= 0, \quad \beta'(x) = 1, \\ \frac{\partial f}{\partial x} &= u(t). \end{aligned}$$

Therefore,

$$\frac{d}{dx}\left(\int_0^x (x-t)\cdot u(t)\,dt\right) = \int_0^x u(t)\,dt$$

3.3 The Adomian Decomposition Method

The decomposition method mostly establishes the solution in the form of a power series. In this method, the solution $u(x)$ will be decomposed into an infinite series of components, given by the series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (2)$$

with $u_0(x)$ identified by all terms out of the integral,

$$u_0(x) = f(x) \quad (3)$$

. Substituting (2) into the Volterra integral equation of the second kind of the form yields:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt$$

which by using a few terms of the series becomes:

$$u_0(x)+u_1(x)+u_2(x)+\ldots = f(x)+\lambda \int_0^x k(x,t)u_0(t)dt+\lambda \int_0^x K(x,t)u_1(t)dt+$$

$$\lambda \int_0^x k(x,t)u_2(t)dt + \lambda \int_0^x k(x,t)u_3(t)dt + ...$$

The components $u_i(x), i > 0$ of the unknown function $u(x)$ are completely determined by using the recurrence manner

$$
\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \lambda \int_0^x K(x,t)u_0(t)dt \\
u_2(x) &= \lambda \int_0^x k(x,t)u_1(t)dt \\
u_3(x) &= \lambda \int_0^x k(x,t)u_2(t)dt
\end{aligned}
$$

and so on.

The above-discussed scheme for the determination of the components $u_i(x), i \geq 0$ of the solution $u(x)$ can be written in a recurrence relation as

$$
\begin{aligned}
u_0(x) &= f(x) \text{ and} \\
u_{n+1}(x) &= \lambda \int_0^x k(x,t)u_n(t)dt, n > 0.
\end{aligned}
$$

The components $u_i(x), i \geq 0$, follow immediately upon integrating the easily computable integrals. With these components determined, the solution $u(x)$ of **VIE** is readily determined in a series form upon using (2). However, for concrete problems, where (2) cannot be evaluated, a truncated series $\sum_{n=0}^{k} u_n(x)$ is usually used to approximate the solution $u(x)$.

Question1: Consider the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)\,dt$$

Solution

It is clear that $f(x) = 1$, $\lambda = 1$, and $k(x,t) = 1$. Using decomposition series solution and the recursive scheme to determine the components u_n, $n \geq 0$:

$$\begin{aligned}
u_0(x) &= 1 \\
u_1(x) &= \int_0^x u_0(t)\,dt = x \\
u_2(x) &= \int_0^x u_1(t)\,dt = \int_0^x t\,dt = \frac{1}{2}x^2
\end{aligned}$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + ...,$$

$$u(x) = 1 + x + \frac{1}{2}x^2 + ...$$

and this converges to the closed form solution

$$u(x) = e^x$$

. **Question 2** :Consider the Volterra integral equation:

$$u(x) = x + \int_0^x (t - x)u(t)dt$$

Solution

$$u_0(x) = x$$

$$
\begin{aligned}
u_1(x) &= \int_0^x (t-x)u_0(t)\,dt \\
&= \int_0^x (t-x)t\,dt \\
&= \int_0^x (t^2 - xt)\,dt \\
&= \left[\frac{t^3}{3}\right]_0^x - x\left[\frac{t^2}{2}\right]_0^x \\
&= \frac{x^3}{3} - \frac{x^3}{2} \\
&= \frac{2x^3 - 3x^3}{6} \\
&= -\frac{x^3}{6}
\end{aligned}
$$

$$u_2(x) = \int_0^x (t-x)\cdot u_1(t)\,dt = \int_0^x (t-x)\cdot \frac{-t^3}{6}\,dt = \frac{x^5}{5!}$$

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + ...$$

in a closed form by

$$u(x) = \sin x$$

. **Question 3**: Consider the Volterra integral equation

$$u(x) = (6x - x^3) + \frac{1}{2}\int_0^x tu(t)\,dt$$

.

Solution

$$u_0(x) = f(x) = 6x - x^3,$$

$$u_1(x) = \frac{1}{2}\int_0^x tu_0(t)\,dt = \frac{1}{2}\int_0^x t(6t - t^3)\,dt = x^3 - \frac{1}{10}x^5$$

$$u_2(x) = \frac{1}{2}\int_0^x t \cdot u_1(t)\,dt = \frac{1}{2}\int_0^x t(t^3 - \frac{1}{10}t^5)\,dt = \frac{1}{10}x^5 - \frac{1}{140}x^7$$

$$u(x) = (6x - x^3) + (x^3 - \frac{1}{10}x^5) + (\frac{1}{10}x^5 - \frac{1}{140}x^7) + ...$$

$$u(x) = 6x(Bycancelling)$$

3.3.1 The Modified Decomposition Method

In Volterra integral equations where the non-homogeneous part $f(x)$ consists of a polynomial that includes many terms, or in the case $f(x)$ contains a combination of polynomials and other trigonometric functions, the modified decomposition method works well. To achieve our goal, decompose the function $f(x)$ into two parts such that $f(x) = f_0(x) + f_1(x)$ where $f_0(x)$ consists of only one term, or if needed, more terms in fewer other cases, and $f_1(x)$ includes the remaining terms of $f(x)$.

$$u(x) = f_0(x) + f_1(x) + \lambda \int_a^x k(x,t)u(t)\,dt \quad (4)$$

Substituting (2) in (4) and expanding

$$u_0(x) + u_1(x) + u_2(x) + ... = f_0(x) + f_1(x) + \lambda \int_0^x k(x,t)u_0(t)dt$$

$$+\lambda \int_0^x k(x,t)u_1(t)\,dt + \lambda \int_0^x K(x,t)u_2(t)\,dt + ...$$

The components $u_i(x)$, $i \geq 0$ of the unknown function $u(x)$ are determined in a modified recurrence relation by assigning $f_0(x)$ only to the components $u_0(x)$, whereas the components $f_1(x)$ will be added to the formula of the component $u_1(x)$.

$$u_0(x) = f_0(x)$$

$$u_1(x) = f_1(x) + \lambda \int_0^x K(x,t) \cdot u_0(t)\, dt$$

$$u_2(x) = \lambda \int_0^x k(x,t) u_1(t)\, dt$$

$$u_3(x) = \lambda \int_0^x k(x,t) u_2(t)\, dt$$

This implies that

$$u_0(x) = f_0(x)$$

,

$$u_1(x) = f_1(x) + \lambda \int_0^x k(x,t) u_0(t)\, dt$$

,

$$u_{n+1}(x) = \lambda \int_0^x k(x,t) u_n(t)\, dt, n \geq 1$$

.

Question 1: Consider the equation $u(x) = \cos x + \sin x - \int_0^x u(t)\, dt$.

Solution

Decompose the function $f(x)$ into $f_0(x) = \cos x$ and $f_1(x) = \sin x$. Consequently, $u_0(x) = \cos x$,

$$u_1(x) =, \sin x - \int_0^x \cos t dt = \sin x - [\sin t]_0^x = \sin x - \sin x = 0$$

So, the other components $u_i(x) = 0$, $i \geq 2$.

$$u(x) = \cos x$$

3.4 The Variational Iteration Method

The method admits the correction function in the form of

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t) \left(Lu_n(t) + N\dot{u}_n(t)\right) - g(t)\, dt, \quad n \geq 0$$

To solve any Volterra integral equation by using this method, first transform the equation to its equivalent ODE where the Leibniz rule should be used. Next, determine λ the zeroth component $u_0(x)$ as indicated

$$u_0(x) = u(0) \quad (1)$$

.

$$u_0(x) = u(0) + xu'(0) \quad (2) \qquad u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0) \quad (3)$$

Here zeroth component $u_0(x)$ can be selected according to the order of the resulted ODE. 1, 2, 3 respectively have the first term, the first two terms, and the first three terms of the Taylor series of $u(x)$ at $x = 0$. The exact solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x)$$

Question1: Solve the Volterra integral equation by using the variational iteration method.

$$u(x) = 1 - x + \int_0^x (x-t)u(t)\,dt \quad (1)$$

Solution

$$u(x) = 1 - x + \int_0^x (x-t)u(t)\,dt$$

Differentiating both sides and using Leibniz:

$$u'(x) = -1 + \int_0^x u(t)\,dt \quad (2)$$

The initial condition $u(0) = 1$ obtained by using $x = 0$ in(1) and hence select $u_0(x) = 1$. The correctional function for (2) is $u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(t) + 1 - \int_0^t u_n(r)\,dr)\,dt$. We select $\lambda = -1$. Select $u_0(x) = 1$ which leads to:

$$u_0(x) = 1$$

$$u_1(x) = 1 - \int_0^x \left(u_0^{'}(t) + 1 - \int_0^t u_0(r)\,dr\right)\,dt = 1 - x + \frac{1}{2!}x^2$$

$$u_2(x) = 1-x+\frac{1}{2!}x^2 - \int_0^x (u_1'(t) - \left(\int_0^t u_1(v)dv\right)\,dt = 1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4$$

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n}(-1)^k\frac{1}{k!}x^k = \lim_{n\to\infty}\left(1 - x + \frac{1}{2!}x^2 - \cdots + (-1)^n\frac{1}{n!}x^n\right) = e^{-x}$$

$$u(x) = e^{-x}$$

Question 2: Solve the Volterra integral equation by using the variational iteration method $u(x) = 2\sin x - \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt$.

Solution:

$u(x) = 2\sin x - \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt$. Differentiating both sides:

$$u'(x) = 2\cos x - \frac{1}{2}x^2 + \int_0^x u(t)dt$$

By using x=0 we find u(0)=0 The variational iteration method admits the use of a correction function for the equation by:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(t) - 2\cos t + \frac{t^2}{2} - \int_0^t u_n(r)dr \right) dt$$

Initial conditions:

$$u_0(x) = 0$$

$$u_1(x) = -\int_0^x \left(u_0^{'}(t) - 2\cos t - \frac{1}{2}t^2 - \int_0^t u_0(r)dr \right) dt$$

$$= 2\sin x - \frac{1}{6}x^3$$

$$u_2(x) = 2\sin x - \frac{1}{6}x^3$$

$$u_2(x) = 2\sin x - \frac{1}{6}x^3 = \int_0^x \left(u_1'(t) - 2\cos t + \frac{1}{2}t^2 - \int_0^x u_1(r)dr \right) dr$$

$$= 2x - \frac{1}{6}x^3 + \frac{1}{120}x^3$$

$$u_3(x) = 2x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

The solution is a series of the form:

$$u(x) = x + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \right)$$

$$u(x) = \lim_{n \to \infty} u_n(x)$$
$$u(x) = x + \sin x$$

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