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2022-2023

STRUCTURAL AND OPTICAL PROPERTIES OF ZnO UNDER SOLVOTHERMAL METHOD

Project Report Submitted To The University Of Kerala In Partial Fulfilment For The Requirement For The Award Of The Degree Of

MASTER OF SCIENCE

IN

PHYSICS

BY

ATHUL S KUMAR

REG NO. 63021126007

DEPARTMENT OF PHYSICS

ST. GREGORIOS COLLEGE

KOTTARAKKARA

2021-2023

Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

DEPARTMENT OF PHYSICS
ST. GREGORIOS COLLEGE
KOTTARAKKARA, KOLLAM.

BONAFIDE CERTIFICATE

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Dr. FERGY JOHN

HEAD OF THE DEPT. OF PHYSICS

ST. GREGORIOS COLLEGE, KOTTARAKARA

Examiners

1. 12/9/2023

2. 12/09/23

Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

Dr RINU SAM S
ASSISTANT PROFESSOR
ST GREGORIOS COLLEGE
KOTTARAKKARA

CERTIFICATE

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Dr. RINU SAM S

(Supervising teacher)

Kottarakkara

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Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

A STUDY ON MATROID THEORY

Dissertation submitted to the University of Kerala, in partial fulfilment of the requirement for the award of the

Degree of Master of science

In

Mathematics

By

ANJU THANKACHAN

Candidate code : 62021126005

Exam code : 62020402

DEPARTMENT OF MATHEMATICS

ST. GREGORIOS COLLEGE KOTTARAKKARA

2023

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Mrs. BEENA G P
Head of the Department
Department of Mathematics
St. Gregorios College
Kottarakkara

Mrs. RESHMA. U
Assistant Professor
Department of Mathematics
St. Gregorios college
Kottarakkara

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KOTTARAKARA

Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

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Dissertation submitted to the University of Kerala, in partial fulfilment of the requirement for the award of the

Degree of Master of science

In

Mathematics

By

ANGEL MARIAM JOHN

Candidate code : 62021126003

Exam code : 62020402

DEPARTMENT OF MATHEMATICS

ST. GREGORIOS COLLEGE

KOTTARAKKARA

2023

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Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

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Mrs. BEENA G P
Head of the Department
Department of Mathematics
St. Gregorios College
Kottarakkara

Mr. NEVIN JOHN
Assistant Professor
Department of Mathematics
St. Gregorios college
Kottarakkara

ST. GREGORIOS COLLEGE
Accredited by
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with A Grade
KOTTARAKARA

Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

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RESEARCH AND PG DEPARTMENT OF COMMERCE

ST. GREGORIOS COLLEGE KOTTARAKARA

(Affiliated to the Kerala University)

2021-2023

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Signed by

Mrs. MELSA C. THOMAS

Research & PG Department of

Commerce

St. Gregorios College,

Kottarakkara

Countersigned by

Dr. SUMI ALEX

Head of the department

Research & PG Department of

Commerce

St. Gregorios College,

Kottarakkara

Place: Kottarakkara

Date:

ST. GREGORIOS COLLEGE
Accredited by NAAC with A Grade
KOTTARAKARA

Dr. SUMI ALEX
PRINCIPAL IN CHARGE
ST. GREGORIOS COLLEGE
KOTTARAKARA

M. Sc. MATHEMATICS

2021 - 2023

A STUDY ON REGULAR SEMIGROUP

A STUDY ON REGULAR SEMIGROUP

Dissertation submitted to the University of Kerala, in partial fulfilment of the requirement for the award of the Degree of Master of science

In

Mathematics

By

AVANI S

Candidate code: 62021126008

Course code :

DEPARTMENT OF MATHEMATICS

ST. GREGORIOS COLLEGE

KOTTARAKKARA

2023

CERTIFICATE

This is to certify that this dissertation is a bonafied record of the work carried out by **AVANI S** under my supervision in partial fulfillments of the requirements for the degree of **Master of science in Mathematics** of the **University of Kerala.**

Mrs. BEENA G P
Head of the Department
Department of Mathematics
St. Gregorios College
kottarakkara

Mrs. ANSLIN T M
Assistant Professor
Department of Mathematics
St. Gregorios College
Kottarakkara

ACKNOWLEDGEMENT

First and foremost we concede the surviving presence and flourishing refinement of almighty god for concealed hand yet substantial supervision althrough the dissertation. I would like to expressed our sincere thanks to **Mrs.ANSLIN T M**, Assistant Professor, Department of Mathematics, St. Gregorios College, Kottarakkara for her inspiring guidence and support to complete this dissertation. I also wish to express my profound thanks to **Mrs. BEENA G P**, Head of the Department and all other teachers of mathematics Department for their constant help throughout the course of this work. I extend our sincere thanks to our librarian and other non - teaching staffs for their co-operation and support. Above all I would like to express my sincere gratitude and thanks to my family members, all my friends and well-wishers for their valuable comments and suggestion and making this work a success.

KOTTARAKKARA AVANI S

Date:

ABSTRACT

Through this project, we present a concise study on a regular semigroup. A regular semigroup is a semigroup S in which every element is regular. That is for each a in S there exist an element x in S suchthat $axa = a$. We analyse the properties related to regular semigroup including treatment of equivalence, congruence and also discuss about structure of regular semigroup and its some classes. Regular semigroups are one of the most studied class of semigroup and their structure is particularly amenable to study via Green's relations.

Contents

Introduction

Regular semigroup are one of the most studied classes of semigroup and their structure is amenable to study via Green's relations.Regular semigroups were introduced by J. A Green in his influential 1951 paper "On the structure of semigroups"; this was also the paper in which Green's relation were introduced. The concept of regularity in a semigroup was adopted by John von Neumann.

Through this project we discuss about the regular semigroup and its general properties, structure and some class of regular semigroups.

The first chapter introduced the basic concepts of semigroup and Green's relation in semigroup.

The second chapter introduces the regular semigroup and its general properties and also discuss the representation of regular semigroup and the concept of strict regular semigroup.

The third chapter introduces a class of regular semigroup named as completely regular semigroup which deals with Clifford decomposition, Clifford semigroup and Band.The fourth chapter discuss other classes of regular semigroups ; Locally inverse, Orthodox, Semiband and its properties. This gives a deep information about regular semigroup.

Chapter 1

Preliminary

In this chapter we give certain basic definitions and examples that we need in the sequel. For the notations and terminologies we, [2],[3],[4]

1.1 Semigroup

Definition 1.1.1. *A binary operation on set S is a map $\cdot : S \times S \to S$. This operation is associative if $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$. A semigroup is nonempty set equipped with an associative binary operation denoted by $(S, .)$.*

Definition 1.1.2. *A semigroup S is said to be a commutative semigroup, $xy = yx$ for all $x, y \in S$.*

Definition 1.1.3. *If S is a semigroup contains an element 1 such that $1.x = x = x.1$ for all $x \in S$, then 1 is called identity of S.*

Note 1.1.4. *The semigroup S with identity 1 is called monoid.*

Result 1.1.5. *Every group is a monoid*

Definition 1.1.6. *If a semigroup S with at least two elements contains an element 0 such that $0x = 0 = x0$ for all $x \in S$, then we say 0 is a zero of S and S is called a semigroup with zero.*

Note 1.1.7. *If S is a Semigroup without 0 then we adjoint an extra element 0 to S we define an operations, $0x = x0 = 0$ and $0.0 = 0$,for all $x \in S$. Then $S \cup \{0\}$ is a semigroup with 0 we define*

$$S^0 = \begin{cases} S & \text{if } S \text{ has already } 0 \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

Note 1.1.8. *If S is a semigroup without identity then we adjoint an extra element 1 to S form a monoid we define, $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$ and $1 \cdot 1 = 1$. Then $S \cup \{1\}$ is a monoid and denote*

$$S^1 = \begin{cases} S & \text{if } S \text{ has already identity} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

Theorem 1.1.9. *A semigroup with zero 0 then 0 is unique.*

Note 1.1.10. *Let $x \neq \phi$ and $T_x = \{\ f : f : x \to x\}$ then T_x is a monoid, where binary operation is composition if mapping. That is if $f, g \in T_x$, $fg : x \to x$ by $x(fg) = (xf)g$, then semigroup T_x is called the full transformation semigroup.*

Definition 1.1.11. *Let S be a semigroup and $T \subseteq S$ then T is said to be a subsemigroup of S if T is closed under multiplication, that is for all*
$a, b \in T \Rightarrow ab \in T$

Definition 1.1.12. *If S is a semigroup containing an element e such that $e^2 = e$, then $\{e\}$ is subsemigroup of S and e is called idempotent of S.*

Definition 1.1.13. *Let S be a semigroup and $A \subseteq S$ then A is said to be a left ideal [right ideal] of S if $SA \subseteq S$ [$AS \subseteq S$].*
. In other words for all $a \in A$, $s \in S$, $sa \in A$ [$as \in A$].

If A is both left and right ideal of S, then A is called an ideal of S.

Definition 1.1.14. *A homomorphism of a semigroup S into a semigroup T is a mapping $\psi : S \to T$ which preserves the property;*

$\psi(xy) = \psi(x)\ \psi(y)$, for all $x, y \in S$.

Proposition 1.1.15. *For an equivalence relation $\mathscr{C}$ on a semigroup S is following are equivalent :*

1. *There is a semigroup operation on $S|\mathscr{C}$ such that the projection $S \to S|\mathscr{C}$ is a homomorphism:*
2. *$\mathscr{C}$ admits multiplication ($a\mathscr{C}\ c$, $b\mathscr{C}\, d \Rightarrow ab\ \mathscr{C}\ cd$):*
3. *$\mathscr{C}$ admits multiplication on left ($a\mathscr{C}\ b \Rightarrow xa\ \mathscr{C}\ xb$) and on the right ($a\ \mathscr{C}\ b \Rightarrow ax\ \mathscr{C}\ bx$).*

Definition 1.1.16. *A congruence on a semigroup S is a equivalence relation $\mathscr{C}$ on S which satisfies the equivalence condition1.1.15 The resulting semigroup $S|\mathscr{C}$ is the quotient of S by $\mathscr{C}$.*
The left congruence is a equivalence relation $\mathscr{C}$ which admits multiplication on the left ($a\ \mathscr{C}\ b \Rightarrow xa\ \mathscr{C}\ xb$) and a right congruence is an equivalence relation $\mathscr{C}$ which admits multiplications on the right ($a\ \mathscr{C}\ b \Rightarrow xa\ \mathscr{C}\ xb$).

Definition 1.1.17. *Let S be a semigroup if S has no ideal other than itself S said to be simple.*

Definition 1.1.18. *S be a semigroup with a zero then S is a 0 - simple with following conditions ;*

1. $S^2 \neq 0$
2. S has no ideals except 0 and itself.

Definition 1.1.19. *A semigroup S is called completely simple if it is a simple and contains a primitive idempotent.*

Note 1.1.20.
- *Left translation of S is a mapping* $\lambda : S \to S$ *such that* $\lambda(xy) = (\lambda x)y$ *for all* $x, y \in S$. *If* λ , μ *are left translations then so is* $\lambda\mu$ *since* $\lambda(\mu(x, y)) = \lambda((\mu x)y = (\lambda(\mu x)y)$ *for all* $x, y \in S$.
- *Right translation of S is a mapping* $\rho : S \to S$ *such that* $(xy)\rho = x(y\rho)$ *for all* $x, y \in S$.
- *The left translation* λ *and right translation* ρ *are linked in case* $x(\lambda y) = (x\rho)y$, *for all* $x, y \in S$.

Definition 1.1.21. *The translation hull of S is the set* $\Omega(S)$ *of all ordered pairs* $(\lambda,\ \rho\ )$ *[called bi translation] of linked left and right translation* λ *and* ρ *of S.*

Note 1.1.22. *If* $(\lambda,\ \rho\ )$, $(\rho,\sigma) \in \Omega(S)$ *then* $x(\lambda(\mu y)) = (x\lambda)\ (\mu y) = (x\rho)\ (\sigma y)$ *for all* $x, y \in S$.*Hence* $\Omega(S)$ *is a semigroup under pointwise operation,*
$(\lambda,\ \rho) = (\lambda\ \mu,\ \rho\sigma)$

Definition 1.1.23. *A bicyclic semigroup is an inversion of semigroup such as monogenic that is, generated by a single element. The Idempotents of the bicyclic semigroup form a chain, which is ordered with respect to the type of positive numbers.*

Definition 1.1.24. *An inverse semigroup is a semigroup S such that gor every element* $s \in S$ *there exists a unique "inverse "* $s' \in S$ *such that* $ss's = s$ *and* $s'ss' = s'$. *It is evident from this that* $s'' = s$

Result 1.1.25. *The idempotents in the inverse semigroup form a subsemigroup which is commutative and idempotent.Then for any idempotents* e, f *we define an order* $\leq$ *on idempotents by* $e \leq f$ *if and only if* $e = ef$.

Proposition 1.1.26. *Let S be a regular semigroup with set E of idempotents and* $e, f \in E$.*Then the set S(e, f), defined by*
$S(e, f) = \{g \in V(ef) \cap E : ge = fg = g\}$ *is nonempty.*

Definition 1.1.27. *The set S(e, f) is called the sandwich set of e and f. It has obvious alternative characterization,* $S(e, f) = \{g \in E : ge = fg = g, egf = ef\}$.

Proposition 1.1.28. *Let e, f and g be idempotents in a regular semigroup.*

1. *If* $e\mathscr{L}f$ *then S(e, g) = S(f, g);*

2. *if* $e\mathscr{R}f$ *then S(g, e) = S(g, f).*

Proposition 1.1.29. *Let e, f be idempotents in a regular semigroup S. Then S(e,f) is a subsemigroup of S and is a rectangular band.*

Theorem 1.1.30. *Let a be an element of a regular D-class D in a semigroup S.*

1. *If* $a' \in V(a)$, *then* $a' \in D$ *and the two* $\mathcal{H}$*-classes* $R_a \cap L'_a$, $L_a \cap R'_a$ *contain, respectively, the idempotents* aa' *and* $a'a$.

2. *If b in D is such that* $R_a \cap L_b$ *and* $L_a \cap R_b$ *contain idempotents* e, f, *respectively, then* $\mathcal{H}_b$ *contains an inverse* a^* *of a such that* $aa^* = e$, $a^*a = f$.

3. *No* $\mathcal{H}$*-class contains more than one inverse of small a.*

Proposition 1.1.31. *If U is a regular subsemigroup of semigroup S, then* $L^U = L^S \cap (U \times U)$, $R^U = R^S \cap (U \times U)$,$H^U = H^S \cap (U \times U)$.

Proposition 1.1.32. *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is completely simple;*

2. *S is completely regular, and, for all x, y in S,* $xx^{-1} = (xyx)(xyx)^{-1}$

3. *S is completely regular and simple.*

Corollary 1.1.33. *If e is an idempotent in a semigroup S, then* H_e *is a subgroup of S. No* $\mathcal{H}_e$ *- class in S can contain more than one idempotent.*

Corollary 1.1.34. *A semigroup S is simple if and only if* $SaS = S$ *for all a in S, that is, if and only if for every a, b in S there exist* x, y *in S such that* $xay = b$.

Lemma 1.1.35. *S is regular, and has exactly two D - classes, namely* $\{0\}$ *and* $D = S|\{0\}$. *If a,b D, then either ab= 0 or ab* $\in R_a \cap L_a$. *The latter occurs if and only if* $L_a \cap R_b$ *contains an idempotent.*

Theorem 1.1.36. *Let S be a semigroup without zero. Then the following conditions are equivalent:*

1. *S is completely simple;*
2. *S is regular, and has the 'weak cancellation' properties : for all a, b, c in S,* $[ca = cb$ *and* $ac = bc] \Rightarrow a = b$
3. *S is regular, and for all a in S,* $aba = a \Rightarrow bab = b$

Definition 1.1.37. *A semilattice is a commutative, idempotent semigroup. That is a commutative band.*

1.2 Green Relation on Semigroup

Definition 1.2.1. *Green's preorder on semigroups are*
$a \leq_{\mathscr{L}} b \Rightarrow a = ub$ *for some* $u \in S^1$

$a \leq_{\mathscr{R}} b \Rightarrow a = bv$ *for some* $v \in S^1$

$a \leq_{\mathscr{H}} b \Rightarrow a = ub = bv$ *for some* $u, v, \in S^1$
$a \leq_{\mathscr{J}} b \Rightarrow a = ubv$ *for some* $u, v \in S^1$
each relation is preorder.

Note 1.2.2.
- *In pre-order* $a \leq b$ *implies a is a multiple of b in same sense.*
- *If S has a zero element and an identity element then* $0 \leq x \leq 1$ *for all* $x \in S$ *under all four pre-order0*
- $a \leq_{\mathscr{L}} b$ *also written* $a \leq_L$, $a \leq b$ $(\mathscr{L})$ *or* $L_a \leq L_b$

Similarly a$\leq_{\mathscr{R}}$ also written a$\leq_R$ b, a$\leq$ b $(\mathscr{R})$ or $R_a = R_b$.
Similarly for $\leq_{\mathscr{H}}$, $\leq_{\mathscr{J}}$.

Definition 1.2.3. *Green's pre-order can be defined in terms of ideals of S. A left ideal of S is a subset L⊆ S such that SL ⊆ L. A right ideal of S is subset R⊆ S such that RS ⊆ R.*

Result 1.2.4. *Green's preorder* $\leq_{\mathscr{H}}$ *induces a partial order on the set of idempotents called the Rees order.*

Proposition 1.2.5. *A partial order on set* $E(S)$ *of idempotents of semigroup S defined by* $e \leq f \iff e \leq_{\mathscr{R}} f \iff ef = fe = e$.

Definition 1.2.6. *Green´s relation on semigroup S are equivalence relation* $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{J}$,

$a\mathscr{L}\ b \iff a\leq_{\mathscr{L}} b$ *and* $b\leq_{\mathscr{L}} a \iff S^1a = S^1b$

$a\mathscr{R}\ b \iff a\leq_{\mathscr{R}} b$ *and* $b\leq_{\mathscr{R}} a \Longrightarrow aS^1 = bS^1$

$a\mathscr{H}\ b \iff a\leq_{\mathscr{H}} b$ *and* $b\leq_{\mathscr{H}} a \iff a\mathscr{L}\ b$ *and* $a\mathscr{R}\ b$

$a\mathscr{J}\ b \iff a\leq_{\mathscr{J}} b$ *and* $b\mathscr{J}\ a \iff S^1aS^1 = S^1bS^1$

Thus $\mathscr{H} = \mathscr{L} \cap \mathscr{R} \subseteq \mathscr{L}$, $\mathscr{R} \subseteq \mathscr{J}$, $\mathscr{L}$ *is right congruence* ($a\mathscr{L}\ b \Rightarrow ac\ \mathscr{L}\ bc$) *since* $S^1a = S^1b \Rightarrow S^1ac = S^1bc$, *dually* $\mathscr{R}$ *is a left congruence* ($a\mathscr{R}\ b \Rightarrow ca\ \mathscr{R}\ cb$).

Definition 1.2.7. *Green´s preorders induces partial order relations on the quotient set* $S|\mathscr{L}$, $S|\mathscr{R}$, $S|\mathscr{H}$, $S|\mathscr{J}$;

by defenition,

$L_a \leq L_b \iff a\leq_{\mathscr{L}} b$

$R_a \leq R_b \iff a\leq_{\mathscr{R}} b$

$H_a \leq H_b \iff a\leq_{\mathscr{H}} b$

$J_a \leq J_b \iff a\leq_{\mathscr{J}} b$

when L_a (R_a, H_a, J_a) *denotes* $\mathscr{L}$ - ($\mathscr{R}$, $\mathscr{H}$, $\mathscr{J}$) *class of a.*

Proposition 1.2.8. *For all* a, $b \in S$

, $a\mathscr{L}\ x\ \mathscr{R}\ b$ *for some* $x \in S \iff a\mathscr{R}\ y\ \mathscr{L}\ b$ *for some* $y \in S$. *Hence the binary* *relation $\mathscr{D}$ defined by*

$$a\mathscr{D}b \Longleftrightarrow a\mathscr{L}x\mathscr{R}b \text{ for } x \in S$$
$$\Longleftrightarrow a\mathscr{R}y\mathscr{L}b \text{ for } y \in S \text{ is an equivalence relation.}$$

Result 1.2.9. *The fifth Green´s relation $\mathscr{D}$ is defined by 1.2.8*

Note 1.2.10. *$\mathscr{D}$ is reflexive, symmetric, transitive since $a\ \mathscr{D}\ b\ \mathscr{D}\ c \Rightarrow a\ \mathscr{L}\ x\ \mathscr{R}\ b\ \mathscr{L}\ y\ \mathscr{R}\ c$ for some $x, y \in S$, $a\mathscr{L}\ x\ \mathscr{L}\ z\ \mathscr{R}\ y\ \mathscr{R}\ c$ for some $z \in S$ and $a\mathscr{D}$.*

Lemma 1.2.11. *For an element a of a semigroup S the following are equivalent :*

1. *a ha an inverse;*
2. *$axa = a$ for some $x \in S$;*
3. *R_a contains an idempotent;*
4. *L_a contains an idempotent ;*

Result 1.2.12. *A element a of S is regular in case it satisfies 1.2.11*

Result 1.2.13. *A $\mathscr{D}$ - class is regular when $\mathscr{D}$ is regular.*

Proposition 1.2.14. *A $\mathscr{J}$ - class J which contains idempotents $e > f$ contains a bicyclic subgroups.*

Proposition 1.2.15. *[Hall 's $\mathscr{J}$ - class theorem] - Let A and B be $\mathscr{J}$ - classes of S such that $A \geq B$ in $S|\mathscr{J}$ and every element of B is regular. For each Idempotent e of A there exist an idempotent f of B such that $f \leq e$.*

Chapter 2

Regular Semigroup

2.1 Regular Semigroup

Definition 2.1.1. *A semigroup S is called regular if for each element a belongs to S there exist x belongs to S suchthat* $a = axa$.

- *In the case there exist* a' *belongs S suchthat both* $a = aa'a$ *and* $a' = a'aa'$ *are satisfed such an element* a' *is called inverse for a; the set of inverse of a is denoted by* $v(a) = \{x \in S : a = axa, x = xax\}$.
- *A regular semigroup is called inverse if each element has unique.*
- *Let V(x) denote the set of all inverse S and E(x) denote the set of all idempotents of S.*

2.2 General Properties

Lemma 2.2.1. *[Lallement's Lemma] Let S be a regular semigroup and* ψ *:*$S \to T$ *be a homomorphism. Every idempotent* $\psi(x)$ *of im* ψ *is the image under* ψ *of an idempotent* $e \leq_{\mathscr{H}} x$ *of S.*

Proof. Assume that $\psi(x)$ is idempotent.Let $y \in v(x^2)$ and $e = xyx$. Then e is idempotent.(since $yxxy = y$) $e \leq_{\mathscr{R}} x$, $e \leq_{\mathscr{L}} x$ and
$\psi(e) = \psi(xyx) = \psi(x^2yx^2) = \psi(x^2) = \psi(x)$. □

Proposition 2.2.2. *Let S be regular semigroup and $\psi : S \to T$ be a homomorphism.Then ker$\psi \subseteq \mathscr{H}$ iff ψ seperates the idempotents of S.*

Proof. Assume that ψ seperates the idempotents of S [if e $\neq$ f in E(S), then $\psi(e) \neq \psi(f)$].
Let $\psi(x) = \psi(y)$ and $x' \in V(x)$.Then, $\psi(x'$y) $= \psi(x'$x)$\in E(T)$. By Lallement's lemma, $\psi(x'$y) $= \psi$(e) for some idempotent $e \leq_{\mathscr{H}} x'$y. Since ψ separates the idempotents of S it follows that $x'\ x = e$ and $x\mathscr{L}$e $\leq_{\mathscr{L}} x'y \leq_{\mathscr{L}}$ y. Dually x$\leq_{\mathscr{R}}$y. Exchanging x and y then yields $x\mathscr{H}y$.Thus ker$\psi \subseteq \mathscr{H}$.The converse is clear sincean $\mathscr{H}$ - class contains at most one idempotent. □

Proposition 2.2.3. *Every regular semigroup S has a smallest group congruence $\mathscr{G}$.*

Proof. : In a regular semigroup, every $\mathscr{L}$ - class and$\mathscr{R}$ - class contains an idempotent.Hence a regular semigroup which contains only one idempotent has only one $\mathscr{H}$ - class and is necessarily a group. Thus a congruence $\mathscr{C}$ is a group congruence iff S/$\mathscr{C}$ contains only one idempotent : by Lallement's lemma, this happens iff E(S) is contained in $\mathscr{C}$ - class.
Therefore is a smallest congruence on S with this property.

□

Proposition 2.2.4. *Every regular semigroup S has a smallest inverse congruence $\mathscr{I}$.*

Proof. A regular semigroup is an inverse semigroup iff its idempotent commute. Hence a congruence $\mathscr{C}$ on S is an inverse congruence iff the idempotents of S/$\mathscr{C}$ commute ; by Lallement's lemma, this happens if and only if $ef\mathscr{C}fe$ for every $e, f \in E(S)$.
Therefore there is a smallest congruence on S with this property. □

Proposition 2.2.5. *An order realation on any regular semigroup S is defined by :*

$$x \leq y \iff (a) x \leq_{\mathscr{R}} y \text{ and } x = ey \text{ for some } e \in E(R_x)$$
$$\iff (b) x \leq_{\mathscr{L}} y \text{ and } x = yf \text{ for some } f \in E(L_x)$$
$$\iff (c) x \leq_{\mathscr{H}} yandx = xy'x \text{ for all } y' \in v(x)$$
$$\iff (d) x \leq_{\mathscr{H}} yandx = xy'xforsomey' \in V(x)$$

If $x \leq y$, then for each $f \in E(R_y) = E(s) \cap (R_y)$ there exists $g \in E(R_z)$ suchthat $g \leq_{\mathscr{H}} f$ and $x = gy$.

Proof. First we show that $(a), (b), (c)$ and (d) are equivalent. Assume (a), $e \leq_{\mathscr{R}} y$ and $x =$ ey for some e $\in E(R_x)$. If $f \in E(R_x)$, then $e \leq_{\mathscr{R}} f$, $fe = e = efe$, e$f \in E(R_e) = E(R_x)$ $ef \leq_{\mathscr{H}}$ e, and $x =$ efy (Since $fy = y$).Thus (1) implies the last part of the statement. Now let $y' \in V(y)$. Then there exists g $\in E(R_x)$ such that $g \leq_{\mathscr{H}} yy'$ and $x =$ gy;hence $x =$ gyy'g$y = xy'x$ .Also $x =$ e$y \leq_{\mathscr{L}} y$, so $x \leq_{\mathscr{H}} y$.Thus (a) implies (c). Clearly (c) implies (d). Next assume (d), $x \leq_{\mathscr{H}} y$ and x = x$y'x$ for some $y' \in V(y)$.Then $x \leq_{\mathscr{R}} y$, $xy' \in E(R_x)$, and $x = (xy')x$. Thus (d) implies (a). Dually, (b), (c), and (d) are equalent.

To prove $\leq$ is an order relation we use (a). Since S is regular, each R_x contains an idempotent e, and then $x = ex$ shows $x \leq x$. If $x \leq y \leq z$, so that $x \leq_{\mathscr{R}} y \leq_{\mathscr{R}} z$ and $x = ey$, $y = fz$ for some e$\in E(R_x)$, $f \in E(R_y)$ Then $e \leq_{\mathscr{R}} f$, $fe = e = efe$, $ef \in E(R_x)$,$x = efz$, and $x \leq z$. □

Result 2.2.6. *The natural order on a regular semigroup is the order relation 2.2.5.*

Proposition 2.2.7. *In regular semi group S :*

1. *if $e, f \in E(S)$, then $e \leq f$ in the natural order if and only if $e \leq f$ in the Rees order ;*
2. *$x \leq e \in E(S)$ implies $x \in E(S)$;*
3. *$x \leq y$ and $x \mathscr{R} y$ (or $x \mathscr{L} y$) implies $x = y$;*
4. *if $x \leq y$, then for each $y' \in V(y)$ there exists $x' \in V(x)$ suchthat $x' \leq y'$.*

Remark. *A regular semigroup is completely semisimple in case all its principal factors are completely simple or completely semi simple.*

Proposition 2.2.8. *A regular semigroup S is completely semisimple if and only if $x \leq y$ and $x\mathscr{D}y$ in S implies $x = y$.*

Proof. Since S is regular its principal factors are not null and are simple or 0 - simple. Hence S is completely semisimple if and only if all nonzero idempotents in the principal factors are primitive ; equivalently, if no $\mathscr{J}$ - class J of S contains idempotents $e > f$.Then the last part of 2.2.5 shows that $x \leq y$ and $x\mathscr{D}y$ implies $x = y$.
If conversly S is not completely semisimple,then 1.1.15 one of its $\mathscr{J}$ - classes contains a bicyclic subsemigroup T of S ; Since T is bisimple, T is contained in a simple $\mathscr{D}$ - class, which then contains idempotents $e > f$. □

Remark. *A regular semigroup is primitive when all its nonzero idempotents are primitive.*

Proposition 2.2.9. *A regular semigroup is primitive if and only if all its nonzero elements are primitive.*

Note 2.2.10.

1. *Since $S/\mathscr{J}$ is directed down, a primitive regular semigroup without zero is completely simple.*

2. *Let $(S_i)_{i\in I}$ be any set of semigroups with zero such that $S_i \cap S_j = O$ whenever $i \neq j$. The 0-direct union of the semigroups S, is the disjoint union $S = (\cup_{i\in I} S_i|0) \cup \{0\}$ in which each S_i is a subsemigroup of S and $S_iS_j = 0$ whenever $i \neq j$. If every S_i is completely 0-simple, then S is primitive.*

Theorem 2.2.11. *A regular semigroup with zero is primitive if and only if it is a 0-direct union of completely 0-simple semigroups.*

Proof. Let S be primitive regular and J be a nonzero $\mathscr{J}$-class of S. By Hall's $\mathscr{J}$ - class Theorem (preposition 1.2.15). $J_x < J$ in $S/\mathscr{J}$ implies $J_x = 0$.
Hence $S'JS' = J \cup \{0\}$ and P_j is a subsemigroup of S. Also $J_a \neq J_b$ implies $J_{ab} \leq J_a, J_b$ and $ab = 0$. Therefore S is the 0-direct union of its principal factors.
□

2.3 Petrich Representation

Definition 2.3.1. *Petrich representation is represents a regular semigroup S by bitranslation of the traces of its $\mathscr{D}$ - classes, so that S can be described in terms of translation hulls of completely simple and 0 - simple semigroup.*

Note 2.3.2. *1. Let S be any semigroup and D be regular $\mathscr{D}$ - class of S with trace $T = T_D = D \cup \{o\}$. Let $*$ denote the multiplication on T.*
Then $a * b = ab$ if $a, b \in D$ and $ab \in R_a \cap L_b$, otherwise $a * b = 0$ for each $s \in S$ and $a \in T$ define $\lambda^s a = \lambda^s_D a = sa$ if $sa \mathscr{L} a \in D$,0 Otherwise.
$a\rho^s = a\rho^s a = as$ if $as \mathscr{R} a \in D$, 0 Otherwise. and $\chi^s = \chi^s_D = (\lambda^s, \rho^s)$.

Lemma 2.3.3. *χ_D is a homomorphism of S into $\Omega(T_D)$.*

Proof. Let $a, b \in T$. If a = 0 or b= 0, then $\lambda^s(a*b) = 0 = (\lambda^s a)*b$. Assume $a, b \in D$. If $(\lambda^s a) * b \neq 0$, then $sa\mathscr{L}a$, $sa * b = sab \in R_{sa} \cap L_b$, $L_a \cap R_b = L_{sa} \cap R_b$ contains an idempotent, $ab \in R_a \cap L_a$, $sab \in R_{sa} \cap L_{ab}$, and $\lambda^s(a * b) = sab = (\lambda^s a) * b$. Conversly assume $\lambda^s(a * b) \neq 0$.Then $a * b = ab \in R_a \cap L_b$ and $sab \in L_{ab} = L_b$. Hence $a = abu$, $ab = vsab$ for some $u, v \in S$, $a = abu = vsabu = vsa$, and $sa\mathscr{L}a$. Then $sab \in R_{sa}$, since $\mathscr{R}$ is a left congruence, and $(\lambda^s a) * b = sab = \lambda^s(a * b)$. Thus λ^s is a left translation of T.
Dually ρ^s is a right translation of T. Let $ab \in D$;assume $(a\rho^s) * b \neq 0$. Then $as\mathscr{R}a$ and $as * b = asb \in R_{as} \cap L_b = R_a \cap L_b$. Hence $sb\mathscr{L}b$ (since $asb\mathscr{L}b$), $asb \in R_a \cap L_{sb}$, and $a*(\lambda^s b) = a*sb = asb = (a\rho^s)*b$. Dually $a*(\lambda^s b) \neq 0$ implies $(a\rho^s) * b = a * (\lambda^s b)$. Therefore λ^s and ρ^s aare linked and $\chi^s = (\lambda^s, \rho^s) \in \Omega(T)$.
Finally let $s, t \in S$. If $\lambda^{st} a \neq 0$,then $sta\mathscr{L}a \in D$, so that $ta\mathscr{L}a$ and $\lambda^s(\lambda^t a) = \lambda^s(ta) = sta = \lambda^{st} a$. If Conversly $\lambda^s(\lambda^t a) \neq 0$,then $a \in D$b, $ta\mathscr{L}a$, and $staa$, so that sta and $\lambda^{st} a = sta = \lambda^s(\lambda^t a)$. Dually $\rho^{st} = \rho^s \rho^t$, and χ is a homomorphism. □

Note 2.3.4. *Given a semigroup T, call a subsemigroup B of $\omega(T)$ bitransitive in case, for all $a, b \in T$, $a\mathscr{L}b$ implies $a = \beta b.$, $b = \beta' a$ for some $\beta, \beta' \in B^1$,and dually $a\mathscr{R}b$ implies $a = b\beta$, $b = a\beta'$ for some $\beta, \beta' \in B^1$.*

Lemma 2.3.5. *χ_D is a bitransitive subsemigroup of $\Omega(T_D)$.*

Definition 2.3.6. *The Petrich representation of S is the homomorphism $\chi : S \to \Pi_{D\in s/\mathscr{D}}\, \Omega(T_D)$ defined by $\chi^s = (\chi^s_D)_{D\in s/\mathscr{D}}$.*

Theorem 2.3.7. *When S is regular, χ is injective homomorphism.*

Proof. Assume $\chi^s = \chi^t$ and let x be an inverse of s. With $D = D_s = D_x$ we have $sx\mathscr{L}x$, $\lambda^s x = sx \in D$, $\lambda^t x = sx \neq 0$ and $tx = sx$. Dually $xs\ \mathscr{R}\ x$ and $xt = x\ \rho^t = x\rho^s = xs$.Then s$\mathscr{R} sx = tx \leq_{\mathscr{R}}$ t. Exchanging s and t yields s $\mathscr{R}$ t.Hence t $\mathscr{R}$ $sx = (sx)^2$ and $t = sxt = sxs = s$. □

Result 2.3.8. *Therorem 2.3.7follows that regular semigroup S is a subdirect product of the semigroup $\chi_D(S)$ by 2.3.5.*

Corollary 2.3.9. *Every regular semigroup S is subdirect product of the semigroups $\chi_D(S)$, each of which is a regular bitransitive subsemigroup of the translational hull of a completely simple or completely 0 - simple semigroup.*

Proof. In a completely 0-simple semigroup, $ab \in L_a \neq O$ implies $ab \in R_a \neq 0$ and conversely. A completely semisimple semigroup S has similar property: if ab in S with $a\mathscr{J}b$, then $b = tab$ for some $t \in S$, $ea = a$ for some idempotent $e \in D$, $b = teab$, $te \in J_a$, since $b \leq_{\mathscr{J}} teab \leq_{\mathscr{J}} te \leq_{\mathscr{J}}$, $ab\mathscr{L} \neq 0$ in the principal factor of $J_a = J_b$, $ab\mathscr{R}a$ in the principal factor, and $ab\mathscr{R}$a in S. Dually, $ab\mathscr{R}$ a implies $ab\mathscr{L}$ b when $a\mathscr{C}$ b. It follows that χ^s_D is innerbitranslation of s when $s \in D$. If D is no the kernal of S, then $\chi^s_D = 0$ for some s and $\chi_D(x)$ is an extension type of T_D.

□

2.4 Strict Regular Semigroup

Definition 2.4.1. *Regular semigroup which are subdirect products of completely simple and 0 - simple semigroups are sometimes called Strict regular semigroups.*

Theorem 2.4.2. *For a regular semigroup S the following conditions are equivalent :*

1. *S is a subdirect product of completely simple and completey 0 - Simple semigroup ;*

2. *for every $\mathscr{J}$ - classes $A \geq B$ and idempotent e$\in$ A there is exactly one idempotent $f \in B$ such that $e \geq f$;*

3. *for every $\mathscr{J}$ - classes $A \geq B$ there is for each $x \in A$ exactly one $y \in B$ such that $x \geq y$.*

Either conditions implies that S is completely semisimple.

Proof. First assume that S is a subdirect product S $\subseteq \Pi_{i\in I}$ Si of completely simple and 0 - simple semigroups $(S_i)_{i\in I}$. Each element a of S is surjective homomorphism S $\longrightarrow S_i$.
Let A and B be $\mathscr{J}$ -clases such that $A \geq B$ in S/$\mathscr{J}$.
Let e$\in$ $E(A) =$ E(S)$\cap A$. By Hall's $\mathscr{J}$ - class theorem (1.2.15), there is idempotent $f\in$ $E(B)$ suchthat $e \geq f$. Suppose that $f, g \in E(B)$ and $e \geq f, g$. Since f $\mathscr{J}$ g we have $f_i \mathscr{J}$ g_i for every i. which in the completely 0 - simple semigroup S_i implies either
$f_i = g_i = 0$ or f_i, $g_i \neq 0$; in the second case $e_i \geq f_i$, g_i implies $f_i = g_i$, since all nonzero idempotents of S_i are primitive.Hence $f = g$ and we have proved that (1) implies (2).

Now assume (2). Let $A \geq B$ be $\mathscr{J}$ - classes ; for each idempotent e$\in$ A denote by $\overline{e}$ the unique idempotent g$\in$ B suchthat $e \geq g$. The following Lemma shows that (3) holds. □

Lemma 2.4.3. *Let $A \geq B$ be $\mathscr{J}$ - classes, $x \in A$, $e \in E(R_x)$, and $z \in B$. Then $\overline{e}$ $\mathscr{R}$ $\overline{e}x$; $ez = z$; implies $\overline{e}z = z$; and $z \leq x$ iff $z = \overline{e}x$.*

Proof. Assume $ez = z$. If $z' \in V(z)$, Then $ez = z$ implies z' e $\in V(z)$ and zz' $e \in E(R_z) \subseteq E(B)$.Then zz' e $\leq e$ yields zz' e $= \overline{e}$ and $\overline{e}z = zz'$ $ez = z$.

Let y $= \overline{e}$ x. Then e $\mathscr{R}$ x implies $\overline{e} = \overline{e}$ e $\mathscr{R}\overline{e}x$, $y \in B$,
$\overline{e} \in R_y$, $y \leq \overline{e} \leq e \leq_{\mathscr{R}} x$ and $y \leq x$. If coversly $z \in B$, $z \leq x$, then by 2.2.5 thereexist $f \in R_z$ suchthat $f \leq e$ and $z = fx$; then $f = z$ and $z = \overline{e}x$. □

Lemma 2.4.4. *Let $A \geq B$ be $\mathscr{J}$ - classes of a strict regular semigroup.*
Let ${\psi_B}^A : A \longrightarrow B$ assign to each $x \in A$ the element $y \in B$ suchthat $y \leq x$, then ${\psi_B}^A$ is a partial homomorphism. Furthermore ${\psi_A}^A$ is identity on A ;
if $A \geq B \geq C$, then ${\psi_C}^B$ o ${\psi_B}^A = {\psi_C}^A$ and $xy = x_1y_1 = {\psi_C}^A(x)\ {\psi_C}^B(y)$ when $A = J_x$, $B = J_y$, $C = J_xy$ and $x \geq x_1 \geq {\psi_C}^A(x)$, $y \geq y_1 \geq {\psi_c}^B(y)$.

Proof. As above denote ${\psi_B}^A(x)$ by $\overline{x}$. By 2.4.2 $\overline{x} = \overline{e}x$ whenever $e \in E(R_x)$ Dually $\overline{x} = x\overline{y}$ whenever $y \in E(l_x)$. Assume $x, y, xy \in A$ and let $e \in E(x)$, $f \in E(R_y)$. Then $xy = xfy$, so $x \leq_{\mathscr{J}} xf \leq_{\mathscr{J}} xfy$ and $xf \in A$.Since P_A is completely simple or 0 - simple by 2.4.3,we have $xf \in L_f$, $xf \in R_x = R_e$, Hence, $\overline{xf} = xf\overline{f} = x\overline{f}$; $\overline{xf} = \overline{e}\overline{xf} = \overline{e}x\overline{f} = \overline{x}\overline{f}$; and $\overline{xy} = \overline{e}xy = \overline{e}xfy = \overline{xf}y = \overline{x}\overline{f}y = \overline{xy}$.
Thus ${\psi_B}^A$ is a partial homomorphism. It is clear that ${\psi_A}^A$ is the identity on A and that ${\psi_C}^B$ o ${\psi_B}^A = {\psi_C}^A$ when $A \geq B \geq C$.Now let $x, y \in S$ and $A = J_x$, $B = J_y$, $C = J_xy$. Let $\overline{s}$ denote ${\psi_C}^J(s)$and take $e \in E(R_x)$, $f \in E(L_y)$.Then $e(xy)$ $= xy = (xy)f$ and $xy = exyf = \overline{e}xy\overline{f} = \overline{xy}$, by 2.4.2.If $x \geq x_1 \geq \overline{x}$, $y \geq y_1 \geq \overline{y}$, then $\overline{x_1} = \overline{x}$, $\overline{y_1} = \overline{y}$ and $xy \geq_{\mathscr{J}} x_1y_1 \geq_{\mathscr{J}} \overline{xy}$ so that $x, y \in C$ and $x_1y_1 = \overline{x_1}\ \overline{y_1} = \overline{x}$ $\overline{y} = xy$ □

Definition 2.4.5. *A Tree is partialy ordered set T in which each principal ideal $\{\ x \in T\ :\ x \leq t\}$ is a chain. Equivalently, $x, y, \leq t$ implies $x \leq y$ or $y \leq x$.*

Definition 2.4.6. *tree T the height h(t) of an element t is number of elements of chain $\{x \in T\ :\ x < t\}$.*

Result 2.4.7. *If t has finite height, then either t is minimal (if h(t) = 0) or there is greatest $x < t$, the predecessor of t.*

Result 2.4.8. *If S is strict regular semigroup and tha $S/\mathscr{J}$ is a tree in which every element has finite height.*

Theorem 2.4.9. *[Lallement and Petrich] The following conditions on a semigroup S with zero are equivalent :*

1. *S is regular and its idempotents form a tree in which every element has finite height ;*
2. *S is a strict regular semigroup and its $\mathscr{J}$ - classes form a tree in which every element has finite height ;*
3. *S is a tree of completely 0 - simple semigroups.*

Proof. It is clear that (2) $\Longrightarrow$ (1). Conversly assume that (1) holds. S is completely semisimple, since a principal factor of S cannot contain a bicyclic subsemigroup with its infinite descending chain of idempotents. S is strict regular (2) of theorem 2.4.2 holds. S/$\mathscr{J}$ is a tree in which every element has finite height. since $A > B$ in S/$\mathscr{J}$ $\Longrightarrow e > f$, for some idempotents e$\in A$, $f \in B$.
Thus (1) $\Longrightarrow$ (2).
It follows from lemma 2.4.4 that (2) $\Longrightarrow$ (3).
Finally let S be a tree of completely 0 - simple semigroup $S_t(t \in T)$, where T is a tree in which every element has finite height. By definition every nonzero product in S_t is a product in S; therefore S is regular. Also an ideal of S which contains $x \in S_i|0$ also contains $\psi_i(x)$ and intersects $S_u|0$ for every $u < t$. Hence the principal ideal generated by $x \in S_i|0$ is $\bigcup_{u \leq t} S_u|0$. Thus the $\mathscr{J}$ - classes of S are the sets $J_t = S_t|0$ and S/$\mathscr{J} \cong T$.
It remains to show that S is a strict regular semigroup. We prove properly (2) in 2.4.2.Let $e \in J_t$ and $u \leq t$. As before there is an idempotent $f \in J_u$ such that $f \leq e$. It remains to show that f is unique. If $t = u$ then $f = e$ since the idempotents of S_t are primitive. Otherwise $t > u$ and we prove the uniqueness of f by introduction on the height of t.
If $f \leq e$, then $f = e = \psi_t(e)f$, $f = fe = f\psi_t(e)$ and $f \leq \psi_t(e)$, where $\psi(e) \leq S_t$ is an idempotent, hence f is a unique by induction hypothesis. □

Chapter 3

Completely Rgular Semigroup

Note 3.0.1. *A group (G, μ) can alternatively be regarded as having three operations, namely the binary operation $\mu : (a, b) \to ab$, the unary operation $a \to a^{-1}$, and the 0-ary operation (the constant) 1. If we wish to emphasize this aspect, we write $G = (G, \mu^{-1}, 1)$. From this point of view, a morphism $\phi : G \to H$ between two groups is defined by the properties*
$(ab)\phi = (a\phi)(b\phi)$, $(a^{-1})\phi$ =$(a\phi)^{-1}$, $1\phi = 1$.

Definition 3.0.2. *A semigroup (S, μ) will be called a U-semigroup if a unary operation $a \to a'$ is defined on S, with the property that $(a')' = a$,for every a in S. We write $S = (S, \mu, \prime)$.*

Result 3.0.3. *Every semigroup may be regarded as a U- semigroup: the most obvious approach is to define $a' = a$ for every a in S.*

Note 3.0.4. *The unary operation must interact in some way with the binary operation. Two versions of interaction are the first, in which a' is usually denoted by $a*$, gives us a $*-$semigroup, or a semigroup with involution ; here the properties of the unary operation are given by*
$(a^*)^* = a, (ab)^* = b^*a^*$.*The second, in which we shall write a' as a^-1,gives us what we shall call an I-semigroup; here the properties are*
$(a^{-1})^{-1} = a, aa^{-1}a = a$.
Since these equations are to hold for every element of S, it follows
$(a^{-1})^{-1} = a, aa^{-1}a = a$, *and so a^{-1} is an inverse of a.*

3.1 The Clifford decomposition

Definition 3.1.1. *A semigroup S will be called completely regular if there exists a unary operation $a \to a'$ on S with the properties*
$(a)^{-1} = a, aa^{-1}a = a, aa^{-1} = a^{-1}a.$

Proposition 3.1.2. *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is completely regular;*
2. *every element of S lies in a subgroup of S;*
3. *every $\mathcal{H}$ - class in S is a group.*

Proof. (1) $\Rightarrow$ (2).Let $a \in S$, and let $aa^{-1} = a^{-1}a = e$. Then, by Theorem 1.1.30, $a \in R_e \cap L_e = H_e$, and H_e is a subgroup of S by Corollary 1.1.33.
(2) $\Rightarrow$ (3). Let $a \in S$; then $a \in G$ for some subgroup G of S. Denote the identity element of G by e, and the inverse of a within G by a^*. Then from
$ea = ae = a$ and $aa^* = a^*a = e$
it follows that $a\mathcal{H}e$, and hence $H_a = H_e$, a group.
(3) $\Rightarrow$ (1).For each a in S, define a^{-1} to be the unique inverse of a within the group H_a. (Notice that the element a may have several inverses in S, but only one of them lies in H_a.) Then it is clear that $(a^{-1})^{-1} = a, aa^{-1}a = a, aa^{-1} = a^{-1}a$, and so S is completely regular. □

Proposition 3.1.3. *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is completely simple;*
2. *S is completely regular, and, for all x, y in S, $xx^{-1} = (xyx)(xyx)^{-1}$.*
3. *S is completely regular and simple.*

Proof. (1) $\Rightarrow$ (2) Let S be completely simple, and for each a in S, let a^{-1} be the unique inverse of a lying inside H_a. Let $x, y \in S$. Then by Lemma 1.1.35, applied

to the case where 0 is indecomposable, we deduce that $xyx\mathcal{H}x$, and it then follows that $xx^{-1} = (xyx)(xyx)^{-1}$, as required.
(2) $\Rightarrow$ (3).Let $a, b \in S$. Then

$$xx^{-1} = (xyx)(xyx)^{-1}$$

and so $J_a \leq J_b$. By interchanging the roles of a and b we may equally well show that $J_b \leq J_a$. It follows that $\mathcal{J} = S$ x S, and so S is simple. (3) $\Rightarrow$ (1). Suppose that S is completely regular and simple. We shall show that every idempotent of S is primitive, from which it will follow, by Theorem 1.1.36, that S is completely simple. Accordingly, let e, f be idempotents in S, and suppose that $f \leq e$, so that $ef = fe = f$. Then,

since S is simple, there exist z, t in S such that $e = zft$. (See Corollary 1.1.34) We now produce 'improved' versions of z and t by defining $x = exf$

and $y = fte$; we still have,

$$xfy = (ezf)f(fte) = e(zft)e = e^3 = e,$$

but now have the extra advantage that $ex = xf = x$ and $fy = ye = y$. Now S is completely regular and so, by Proposition 3.1.2, the element x belongs to H_g for some idempotent g.Thus $gx = xg = x$, and there exists x^-1 in H_g, such that $xx^{-1} = x^{-1}x = g$. As a consequence, $gf = x^{-1}xf = xx^{-1} = g$. But we also have

$$gf = gef = gxfyf = xyf = ef = f,$$

and so $g = f$.Hence

$$f = fe = ge = gxfy = xfy = e.$$

We have shown that $f \leq e$ implies fe for every pair of idempotents in S. Thus every idempotent in the non-empty set of idempotents of S is primitive, and so S is completely simple as required. □

Theorem 3.1.4. *Every completely regular semigroup is a semilattice completely simple semigroups.*

3.2 Clifford semigroup

Definition 3.2.1. *A Clifford semigroup is defined as a completely regular semigroup (S,,-1) in which, for all z, y in S*
$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1})$.

Note 3.2.2. *In an arbitrary semigroup S, let us say that an element c is central if cs = sc for every s in S. The set of central elements forms a subsemigroup of S, called the centre of S.*

Theorem 3.2.3. *Let S be a semigroup with set E of idempotents. Then the following statements are equivalent:*

1. *S is a Clifford semigroup;*
2. *S is a semilattice of groups;*
3. *S is a strong semilattice of groups;*
4. *S is regular, and the idempotents of S are central;*
5. *S is regular, and* $\mathcal{D}^S \cap (E \times E) = 1_E$.

Proof. (1) ⇒ (2). Let S be a Clifford semigroup. Then S is completely regular, and so is a semilattice Y of completely simple semigroups S. Now every idempotent e in S is expressible as xx^-1 for some x the obvious choice for x is e itself and so the condition in above defenition (3.2.1) says that idempotents commute. This happens within each of the components S_α and so each S_α, being a completely simple semigroup in which idempotents commute, is a group. Thus S is a semilattice of groups.
(2) ⇒ (3). For each α in Y let e_α be the identity element of S_α $(a \in Y)$. Suppose now that $\alpha \geq \beta$. Then for each α in S_α the product $e_\beta a_\alpha$, belongs to $S_{\alpha\beta} = S_\beta$, and so it makes sense to define a map $\phi_{\alpha,\beta} : S_\alpha S_\beta$ by the rule that $a_\alpha \phi_{\alpha,\beta} = e_\alpha a_\alpha$. It is clear that $\phi_{\alpha,\beta}$ is the identity map on $S\alpha$.Also $\phi_{\alpha,\beta}$, is a morphism. To see this, notice that for every a_α, b_α in S_α,
$(a_\alpha \phi_\alpha, \beta)(b_\alpha \phi_\alpha, \beta) = (e_\beta a_\alpha)(e_\beta b_\alpha) = ((e_\beta a_\alpha)e_\beta)b_\alpha$.

Now $e_\beta a_\alpha \in S_\beta$ and e_β is the identity of
S_β. So $(a_\alpha\phi_{\alpha,\beta})(b_\alpha\phi_{\alpha,\beta}) = e_\beta a_\alpha b_\alpha = (a_\alpha b_\alpha)\phi_{\alpha,\beta}$,
as required.
Next, suppose that $\alpha \geq \beta \geq \gamma$ and notice, by a standard property of group morphisms, that, for all α in S_α,

$$\begin{aligned}(a_\alpha\phi_{\alpha,\beta})\phi_{\beta,\alpha} &= e_\gamma(e_\beta a_\alpha)\\ &= (e_\gamma e_\beta)a_\alpha = (e_\beta\phi_{\beta\gamma})a_\alpha\\ &= e_\gamma a_\alpha = a_\alpha\phi_{\alpha,\gamma}\end{aligned}$$

thus $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ as required.
Finally, notice that, for arbitrary α and β in Y and for elements a_α in S_α and b_β in S_β, the product $a_\alpha b_\beta$ lies in S_γ, where $\gamma = \alpha\beta$. Hence

$$\begin{aligned}a_\alpha b_\beta &= e_\gamma(a_\alpha b_\beta) = (e_\gamma a_\alpha)b_\beta\\ &= ((e_\gamma a_\gamma)e_\gamma)b_\gamma\\ &= (e_\gamma a_\alpha)(e_\gamma b_\beta) = (a_\alpha\phi_{\alpha,\gamma})(b_\beta\phi_{\beta,\gamma}\end{aligned}$$

and so S is indeed isomorphic to the strong semilattice of groups $S[Y; S_\alpha; \phi_{\alpha,\beta}]$
(3) $\Rightarrow$ (4) Certainly every strong semilattice of groups $S[Y; G_\alpha; \phi_{\alpha,\beta}]$ is a regular semigroup. Its idempotents are the identity elements e_α, of the groups G_α, and it is easy to calculate that, for all β in Y and all g_β in G_β,
$e_\alpha g_\beta = (e_\alpha\phi_{\alpha,\alpha\beta})(g_\beta\phi_{\beta,\alpha\beta)} = e_{\alpha\beta}(g_\beta\phi_{\beta,\alpha\beta}) = g_\beta\phi_{\beta,\alpha\beta}$,
$g_\beta e_\alpha = (g_\beta\phi_{\beta,\alpha\beta})(e_\alpha\phi_{\alpha,\alpha\beta}) = (g_\beta\phi_{\beta,\alpha\beta})e_{\alpha\beta} = g_\beta\phi_{\beta,\alpha\beta}$;
thus idempotents are central.
(4) $\Rightarrow$ (5) Suppose that $e\mathcal{D}^S f$, where e and f are idempotents. Then, by Theorem 1.1.30 there exists an element a and an inverse a' of a such that $aa{\cdot}a'af$. Hence, using the centrality of the idempotents e and f, we have

$$e = e^2 = a(a'a)a' = afa' = faa' = a'aaa'$$
$$= aa'e = aea = aa'aa = f^2 = f.$$

and we deduce that $\mathcal{D}^S \cap (E{\times}E) = 1_E$

(5) $\Rightarrow$ (1). Each $\mathcal{D} - classcontainsasingleidempotent, andsoisagroup.Thus$D = H $, andsoeachelementahasexactlyoneinverse$a^{-1}, with the properties,

$$(a^{-1}) = a, aa^{-1} = a, aa^{-1} = a^{-1}a.$$

Thus S is completely regular, and so is a semilattice Y of completely simple semigroups S_α. Now for all x, y in S_α we have $xy \in R_x \cap L_y$, and so xDy. Thus each S_α, is contained in a single $\mathcal{D}$ - class, and so has a single idempotent. Hence each S_α is a group.

From (2) $\Rightarrow$ (3) we now deduce that S is a strong semilattice of groups $S[Y; S_\alpha; \phi_{\alpha,\beta}]$,and it then follows easily that for an arbitrary x in S_α and y in S_β,

$$xx^{-1}yy^{-1} = e_\alpha e_\beta = e_{\alpha\beta} = e_\beta e_\alpha = yy^{-1}xx^{-1}$$

Thus S is a Clifford semigroup. □

3.3 Band

Note 3.3.1. *Let B be a band, Since B is completely regular, it decomposes by Theorem 3.1.4 in to a semilattice Y of completely simple semigroups* $S_\alpha(\alpha \in Y)$. *Each of these completely simple semigroups, being a subsemigroup of B, is a band, and it is a band satifying the law* $(xyx)(xyx)' = xx'$, *by Proposition 1.1.31. Since* $x = x'$ *for every* x *in a band, this identity reduces to* $xyx = x$, *and so we conclude that each* S_α, *is a rectangularband.*

Theorem 3.3.2. *Every band is a semilattice of rectangular bands.*

Remark. *Every rectangular band is isomorphic to a cartesian product* $I \times \Lambda$ *with multiplication given by,*

$$(i, \lambda)(j, \mu) = (i, \mu)$$

Proposition 3.3.3. *If* ϕ *is a morphism from a rectangular band* $I_1 \times \Lambda_1$ *into a rectangular band* $I_2 \times \Lambda_2$ *, then there exist maps of* $\phi^l : I_1 \longrightarrow I_2$ *and*
$\phi^r : \Lambda_1 \longrightarrow \Lambda_2$ *such that, for all* (x_1, ξ) *in* $I_1 x \Lambda_1$*,*

$$(x_1, \xi_1)\phi = (x_1\phi^l, \xi_1, \phi^r) \tag{3.1}$$

Conversely, if $I_1 \times I_2$ *and* $\Lambda_1 \times \Lambda_2$ *are arbitrary maps, then the formula (3.1) defines a morphism from* $I_1 \times \Lambda_1$ *into* $I_2 \times \Lambda_2$*.*

Proof. Let $\phi : I_1 \times \Lambda \longrightarrow I_2 \times \Lambda$ be a morphism. Choose a fixed λ_1 in Λ, and for every x_1 in f_1 define a by $x_1\phi^l$.

$$(x_1, \lambda_1)\phi = (x\phi^l, \lambda_2)$$

Similarly, choose a fixed i in I_1, and for every ξ_1 in Λ define $\xi_1\phi^r$ by

$$(i_1, \xi)\phi = (i_2, \xi_1\phi^r).$$

then for all (x_1, ξ_1) in $I_1 \times \Lambda_1$,

$$\begin{aligned}(x_1, \xi_1)\phi &= [(x_1, \lambda_1)(i_1, \xi_1)]\phi = [(x_1, \lambda_1)\phi][(i_1, \xi_1)\phi] \\ &= (x_1\phi^l, \lambda_2)(i_2, \xi_1\phi^r) = (x_1\phi^l, \xi_1\phi^l).\end{aligned}$$

Conversly, if ϕ is defined by (3.1) then, for all (x_1, ξ), (y_1, η) in $I_1 \times \Lambda$,

$$\begin{aligned}[(x_1, \xi_1)(y_1, \eta_1)]\phi = [x_1, \eta_1]\phi = [x_1\phi^l, \eta_1, \phi^l] &= (x_1\phi^l, \xi_1\phi^r)(y_1\phi^l, \eta_1\phi^r) \\ &= [(x_1, \xi_1)\phi][(y_1, \eta_1)\phi].\end{aligned}$$

Thus ϕ is a morphism. □

Corollary 3.3.4. *Let* L_1, L_2 *be left zero semigroups and let* R_1, R_2 *be right zero semigroups. If* ϕ *is a morphism from the rectangular band* $L_1 \times R_1$ *into the* *rectangular band* $L_2 \times R_2$*, then there exist morphisms* $\phi^l : L_1 \longrightarrow L_2$*,* $\phi^r : R_1 \longrightarrow R_2$ *such that*

$$(l_1, r_1)\phi = (l_1\phi^l, r_1\phi^r) \tag{3.2}$$

for all (l_1, r_1) *in* $L_1 \times R_1$*.*
Conversely, for every pair of morphisms $\phi^l : L_1 \longrightarrow L_2$*,* $\phi^r : R_1 \Longrightarrow R_2$*, the formula (3.2) defines a morphism from* $L_1 \times R_1$ *into* $L_2 \times R_2$*.*

Lemma 3.3.5. *If a and b are elements in a regular semigroup S, then*

$$[\lambda_a = \lambda_b and \rho_a = \rho_b] \Rightarrow a = b.$$

Proof. Suppose that $\lambda_a = \lambda_b$ and $\rho_a = \rho_b$, and let $a' \in V(a)$,$b' \in V(b)$. Then

$$a = aa'a = (\lambda_a, a')a = (\lambda_b a') = ba'a,$$

and so $R_a \leq R_b$. Similar arguments show that $L_a \leq L_b$, $R_b \leq R_a$, $L_b \leq L_a$, and so $a\mathscr{H}b$. By Proposition 2.4.1 we may now Suppose that a' and b' have been chosen so that $aa' = bb'$ and $a'a = b'b$, and it then easily follows that $a = ba'a = bb^{=}b$.

□

Chapter 4

Other Classes of Regular Semigroup

4.1 Locally Inverse Semigroup

Definition 4.1.1. *In every regular Semigroup S the subset eSe is clearly a subsemigroup for every idempotent e. It is even a regular subsemigroup, since for every* $x = ese$ *in eSe and every inverse* x' *of* x*,*
$x = xx'x = (xe)x'(ex) = x(ex'e)x.$
$AregularsemigroupSwithsetEofidempotentswillbecalledlocallyinverseifeSeisaninversesemi$

Note 4.1.2. *If a, b are elements of a regular semigroup S with set E of idempotents, then we* $a \leq b$ *if* $R_a \leq R_b$ *and* $(\exists e \in E \cap R_a)a = eb$*.*

Proposition 4.1.3. *Let S be a regular semigroup with set E of idempotents then the relation* $\leq$ *define by (4.1.2) is a partial order relation. Within E the order coincides with the natural order among idempotents :*
$e \leq f$ *if and only if* $ef = fe = e$*.*

Proof. It is clear that $a \leq a$ for every a in S simply choose $e = aa'$. Suppose now that $a \leq b$ and $b \leq a$. Then certainly a R b. Also, there exists idempotents e, f in $R_a = R_b$ such that $a = eb$ and $b = fa$. Since e R f we have $fe = e$ and it then easily follows that $a = eb = feb = fa = b$

To show that $\leq$ is transitive suppose that $a \leq b$ and $b \leq c$. Certainly $R_a \leq R_b \leq R_c$, and there exists e in $E \cap R_a$ and f in $E \cap R_b$ such that $a = eb$ and $b = fc$. Now $R_e = R_a \leq R_b = R_f$, and so $fe = e$. Hence $(ef)^2 = e(fe)f = e^2 f = ef$.
We now have $a = (ef)c$ and from $R_a = R_{efc} \leq R_{ef} \leq R_e = R_a$.
We have that $ef \in E \cap R_a$.
To prove the final assertion, observe that for all e, f in E, $e \leq f$ if and only if $R_e \leq R_f$ and there exists i in $E \cap R_e$ such that $e = if$, that is, if and only if $fe = e$ and $ef = e$. □

Theorem 4.1.4. *Let a, b be elements of a regular semigroup S with set E of idempotents. Then the following statements are equivalent :*

1. $a \leq b$;
2. $a \in bS$ *and* $(\exists a' \in V(a))a = aa'b$;
3. $H_a \leq H_b$ *and* $(all b' \in V(b))a = ab'a$;
4. $H_a \leq H_b$ *and* $(\exists b' \in V(b))a = ab'a$.

Proof. (1) $\Longrightarrow$ (2) is clear, since $e \in E \cap R_a$ if and only if there exists a' in V(a) such that $aa' = e$.
(2) $\Longrightarrow$ (3). We are supposing that $a = bu$ for some u in S, and that $a = (aa')b$. Clearly we take e as aa'. Now notice that $(ua'b)^2 = ua'bua'b = ua'aa'b = ua'b.$.
So define f as $ua'b$, and observe that
$bf = bua'b = aa'b = a$.
(3) $\Longrightarrow$ (4). Suppose that $a = eb = bf$, with $e, f \in E$. Then $R_a \leq R_b$ and $L_a \leq L_b$, and so $H_a \leq H_b$. Also, for every b' in V(b), $ab'a = ebb'bf = ebf = a$.
(4) $\Longrightarrow$ (5) is clear.
(5) $\Longrightarrow$ (1).Suppose that $H_a \leq H_b$ and that there exists an inverse b' of b for which $a = ab'a$. Certainly $R_a \leq R_b$. For every inverse a' of a we see that
$a(a'ab')a = ab'a = a$ and $(a'ab')a(a'ab') = a'(ab'a)a'ab' = a'ab'$;
hence $a'ab' \in V(a)$. Let $e = aa'ab'$; then $e \in E \cap R_a$. From $L_a \leq L_b$ we deduce that a = ub for some u in S. Then $eb = aa'ab'b = ab'b = ubb'b = ub = a$. □

Remark. *It is consequences of above theorem that order $\leq$ can be defined also by the left / right duals of the one - sided definitions. Thus for example, $a \leq b$ if and only if $L_a \leq L_b$ and ($\exists e \in E \cap L_a$.)*

Result 4.1.5. *In an inverse semigroup S the order relation $\leq$ is compatible with the multiplication : $a \leq b$ and $c \in S \Longrightarrow ca \leq cb$ and $ac \leq bc$.*

Theorem 4.1.6. *Let S be a regular semigroup with set E of idempotents. Then the following statements are equivalent :*

1. *S is locally inverse ;*

2. *$\leq$ is compatible ;*

3. *$|S(e, f)| = 1$ for all e, f in E.*

Proof. (1) $\Longrightarrow$ (2). Let $a \leq b$ and let $c \in S$. Thus $R_a \leq R_b$, and there exists e in $E \cap R_a$ such that a = eb. Let a' in V(a) be such that $aa' = e$, choose c' in V(c), and let g be an element of the sandwich set $S(a'a, cc')$. (Thus $ga'a = cc'g = g$ and $a'agcc' = a'acc'$.) Also $v'ga' \in (Vac)$ by Preposition 1.1.29 ,and so the element $f = acc'ga' \in E \cap R_ac$.
Also $f(bc) = acc'ga'bc = aga'bc = aga'aa'bc = aga'ebc = aga'ac = agc = ac.$
We must now show that $R_{ac} \leq R_{bc}$.From $R_a \leq R_b$ we deduce that $a = bu$ for some u in S. Hence for all b' in V(b)
we have $(b'a)^2 = b'ab'a = b'ebb'bu = b'ebu = b'ea = b'a$; thus $b'a \in E$. Moreover, $b'b.b'a = b'a, b'a.b'b = b'ebb'b = b'eb = b'a$, and so $b'a \leq b'b$.
From $a = bu = bb'bu = bb'a$ we deduce that $a\mathscr{L}b'a$, and it follows that there exists an inverse $a''a = b'a \leq b'b$. Also from

$$a = bu = bb'bu = bb' \tag{4.1}$$

we deduce a $\mathscr{L}b'a$, and it follows that there exists an inverse a'' of a such that a''a $= b'a$. To summarize, we now have

$$a''a = b'a \leq b'b \tag{4.2}$$

Also from (4.1) we deduce that

$$a = ba''a. \tag{4.3}$$

As before, let $c' \in V(c)$, and let $h \in$ S(a'', a, cc'). Then from (4.2) we have
(a$''ah)^2 = a''$ a (ha$''a)h = a''ah^2 = a''ah$,
(b'bh)$^2 = b'bha''ab'bh = b'bha''ah = b'bh = b'bh$,
and so $a''ah, b'bh \in E$. In fact $a''ah = a''aha''a = b'ba''aha''ab'b \in b'bSb'b$,
$b'bh = b'bha''a = b'bha''ab'b \in b'bSb'b$,and so both $a''ah$ and $b'bh$ are idempotents within the inverse semigroup $b'bSb'b$. We deduce that
$a''ah = a''aha''aha''aha''ab'bh = (a''ah)(b'bh) = (b'bh)(a''ah) = b'bh$. Finally, denoting the idempotent $c'ha''acbyf$, we conclude, using (4.3), that
$(bc)f = bcc'ha''ac = bhc = bb'bhc = ba''ahc = ahc = ac$, and so $R_{ac} \leq R_{bc}$ as required.
(2) $\Longrightarrow$ (3).Let $g, h \in S(e, f)$, where $e, f \in E$. Then in particular $fg = g$ and so $(gf) = g(fg)f = g^2f$. Moreover, $f(gf) = gf, (gf)f = gf$. and so $gf \leq f$. Similarly $eg \in E$ and $eg \leq e$. By compatibility we deduce that $gh = g(fh) = (gf)h \leq fh = h$, $hg = (he)g = h(eg) \leq he = h$.
That is,$(gh)h = h(gh) = gh$, $(hg)h = h(hg) = hg$, and so $gh = hg$. However, by Proposition 1.1.29, $S(e, f)$ is a rectangular band.
Hence $g = ghg = g^2h = gh = hg = h(hg) = hgh = h$.
We conclude that $|S(e, f)| = 1$.
(3) $\Longrightarrow$ (1).Let $e \in E$, let $a \in eSe$, and let $a' \in V(a) \cap ese$. Then $a'a \in S(a'a, e)$, for $a'aa'a = a'a$, $ea'a = a'a$ and $a'a(a'a)e = a'ae$. Hence in fact, by our assumption, $a'a$ is the only element in $S(a'a, e)$. By the same token, if a'' is another inverse of a in eSe then $S(a''a, e) = a''a$. But, by Proposition 1.1.28, $S(a'a, e) = S(a''a, e)$, and so it follows that $a''a = a'a$. Similarly, by considering $S(e, aa')$ and $S(e, aa'')$, we deduce that $aa'' = aa'$, and it now follows that $a'' = a''aa'' = a'aa'' = a'aa' = a'$.
Hence eSe is an inverse semigroup.

4.2 Orthodox Semigroup

Definition 4.2.1. *A semigroup is called orthodox if it is regular and if its idempotents form a subsemigroup.*

Theorem 4.2.2. *Let S be a regular semigroup with set E of idempotents. Then the following statements are equivalent :*

1. *S is orthodox :*

2. *($\forall e, f \in E$) $fe \in S(e, f)$;*

3. *($\forall a, b \in S$) $V(b)V(a) \subseteq V(ab)$;*

4. *($\forall e \in E$) $V(e) \subseteq E$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that S is orthodox, let $e, f \in E$, and let $g = fe$. Then $ge = fg = g, egf = (ef)^2 = ef$, and so g $= f \in S(e, f)$ by 1.1.28.
(2) $\Longrightarrow$ (3). Let $a, b \in S$ and let $a' \in V(a)$, $b' \in V(b)$. Then by preposition 1.1.28. $b'ga' \in V(ab)$ for all g in $S(a', a, bb')$ From (2) it thus follows that
$b'a' = b'(bb'a'a)a' \in V(ab)$, exactly as required.
(3) $\Longrightarrow$ (4).Let $e \in E$ and let x be an inverse of e: $xex = x, exe = e$. Now both x and ex are idempotents, and so each is each is an inverse of itself. By (3) we deduce that $(ex)(xe)$ is an inverse of $(xe)(ex)$, that is to say, that ex^2e is an inverse of $xe^2x = xex = x$. Hence $x = x(ex^2e)x = (xex)(xex) = (xex)^2 = x^2$, and so x is idempotent as required.
(4) $\Longrightarrow$ (1).Let $e, f \in E$. By preposition 1.1.26 there exist an idempotent g in V(e, f) (an element of the sandwich set S(e, f).But then ef, being an inverse of the idempotent g, must itself be idempotent. Hence S is orthodox. □

Proposition 4.2.3. *Let S be an orthodox semigroup with set E of idempotents. For all a in S, e in E and a' in V(a), the elements aea' and $a'ea$ are idempotent.*

Proof. With the given notation,
$(aea')^2 = aea'aea' = aea'aea'aa' = a(ea'a)^2a' = aea'aa' = aea'$. Thus aea' is an idempotent. This proof for $a'ea$ is similar.

Note 4.2.4. *The set E of idempotents in an orthodox semigroup S forms a band under multiplication and this is expressible as a semilattice Y of of rectangular bands $E_\alpha(\alpha \in Y)$. Certainly $E_\alpha \cap E_\beta \neq \phi$ if $\alpha \neq \beta$, and we also have*

$$E_\alpha E_\beta \subseteq E_{\alpha\beta}, (\alpha, \beta \in Y) \qquad (a)$$

Each E_α is a J^E -class, and it will be consistent with our previous notation to write ${J^E}_e$ for the rectangular band E_α containing e. The formula (a) translates to

$${J^E}_e {J^E}_f \subseteq {J^E}_{ef} = {J^E}_{fe}(e, f \in E).$$

The equivalence J^E is the minimum semilattice congruence on E. From 4.2.2 we know that $V(e) \subset E$ for every e in E. In fact, if $f \in V(e)$ then $efe = e, fef = f$, and it is clear that $f \in {J^E}_e$. Conversly, if f belongs to the rectangular band ${J^E}_e$ then certainly $f \in V(e)$, since any 2 elements of a rectangular band are mutually inverse.
Hence $V(e) = {J^E}_e$, $(e \in E)$.
Thus $V(e)$ is determined solely by nature of a band E.

Proposition 4.2.5. *Let a S, an orthodox semigroup with band E of idempotents. If a' is an inverse of a, then $V(a) = {J^E}_{aa'}a'{J^E}_{aa'}$.*

Proof. Let $e \in {J^E}_{a'a}$ and $f \in {J^E}_{aa'}$.Then $a'aea'a = a'a, aa'faa' = aa'$, and so

$$a(ea'f)a = aa'aea'aa'aa'faa'a = a(a'aea'a)a'(aa'faa')a = aa'aa'aa'a = a,$$

and

$$(ea'f)a(ea'f) = ea'aa'faa'aa'aea'aa'f = ea'(aa'faa')a(a'aea'a)a'f$$

Thus ${J^E}_{a'a}a'{J^E}_{aa'} \subseteq V(a)$.

Conversely, suppose that $a^* \in V(a)$. Then

$$a^* = a^*aa^* = a^*aa'aa^*. \tag{4.4}$$

Now, from
$(a^*a)(a'a)(a^*a) = a^*(aa'a)a^*a = a^*aa^*a = a^*a$ and $(a'a)(a^*a)(a'a) = a'(aa^*a)a'a = a'aa'a = a'a$ we deduce that $a^*a \in {J^E}_{a'a}$. A similar argument shows that $aa^* \in {J^E}_{aa'}$, and it is now immediate from (4.4) that $V(a) \subseteq {J^E}_{a'a}a'{J^E}_{aa'}$. □

Theorem 4.2.6. *A regular semigroup S is orthodox if and only if* $(\forall a, b \in S)[V(a) \cap V(b) \neq 0 \Longrightarrow V(a) = V(b)]$.

Proof. Suppose first that S is orthodox, and that a, b in S are such that $x \in V(a) \cap V(b)$. Then a and b both belong to $V(x)$ and so, by Theorem 1.1.30, xaR^sxb and axL^sbx. Now $xa, xb, ax, bx \in E$, and so,and so, by Proposition 1.1.31, xaR^Exb and axL^Ebx. Certainly xaJ^Exb and axJ^Eb, and so $V(a) = {J^E}_{xa}x{J^E}_{ax} = {J^E}_{xb}x{J^E}_{bx} = V(b)$.Conversely, suppose that S is regular and that we have the given impli cation. Let $e, f \in E$ and let $g \in S(e, f)$. Then from $ge = g$ we may deduce that eg is idempotent. Also $g(eg)g = g$, $(eg)g(eg) = eg$, and so we have that $g \in V(g) \cap V(eg)$. From our assumption we deduce that $V(g) = V(eg)$. Hence in particular $ef \in V(eg)$, and so $ef = (ef)(eg)(ef) = (ef)(efg)(eg) = (eg)^2$. Thus S is Orthodox. □

Result 4.2.7. *The equivalence relation* $\gamma = (x, y) \in SxS : V(x) = V(y)$ *on an orthodox semigroup S turns out to be a congruence.*

Theorem 4.2.8. *Let S be an orthodox semigroup with set E of idempotents. Then the equivalence* γ *defined by 4.2.7 is the smallest inverse semigroup congruence on S. Moreover, for each a in S and each* a' *in* $V(a)$, $a\gamma = {J^E}_{aa'}a{J^E}_{a'a}$.

Proof. To show that γ is a congruence, consider (a, b) in γ and let $c \in S$. Then, for every z in V(a) (= V(b)) and for every c' in V(c), we have $xc' \in V(ca) \cap V(cb)$. Hence V(ca) = V(cb) by Theorem 4.2.6. A similar = argument shows that V(ac) = V(bc), and so y is a congruence.The quotient $S|\gamma$ is certainly regular. By

Lallement's Lemma each idempotent of S/γ is of the form $e\gamma$, where e is an idempotent of S. Now, for any two idempotents e, f in E,

$$V(ef) = J^E{}_{ef} - J^E{}_{fe}$$
$$= V(fe),$$

and from this we deduce that $(e\gamma)(f\gamma) = (f\gamma)(e\gamma)$ in S/γ.
Finally, to show that γ is the least inverse semigroup congruence, let ρ be a congruence on S such that S/ρ is an inverse semigroup, let $(a, b) \in \gamma$, and let $x \in V(a)(= V(b))$. Then both $a\rho$ and $b\rho$ are inverses of $x\rho$ in the inverse semigroup S/ρ, and so $a\rho = b\rho$. We have shown that $\gamma \subseteq \rho$.
To prove the final statement of the theorem, suppose that $b \in a\gamma$. Then V(a) = V(b), and so $a' \in V(b)$ for every a' in V(a). It now follows from Proposition 4.2.5 that

$b \in V(a') = J^E{}_{aa'} a J^E{}_{a'a}$.

Conversely, if $b \in J^E{}_{aa'} a J^E{}_{a'a} = V(a')$, then $V(a) \cap V(b) \neq \phi$, and so V(a) = V(b) by Theorem 4.2.6. Thus $b \in a\gamma$, as required. □

4.3 Semiband

Definition 4.3.1. *A regular semigroup generated by its idempotents is called a semigroup.*

Note 4.3.2.

1. *Semibands differ from locally inverse and Orthodox semigroups in the sense that they are not generalization of inverse semigroups.*

2. *A regular semigroup is orthodox and a semiband if and only if it is a band, and it is both an inverse semigroup and a semiband if and only if it is a semilattice.*

3. *Consider the set $Sing_n$, of all singular maps from the set $[n] = \{1, 2, ..., n\}$ into itself. (By a singular map we mean one that is not a bijection.) This is a finite semigroup, of order $n^2 - n!$.*

Theorem 4.3.3. *For all $n \geq 2$, the semigroup $Sing_n$, is a semiband.*

Proof. To show that $Sing_n$, is regular, let a $Sing_n$, and define $\varepsilon : [n] \longrightarrow [n]$ as follows: if $j \in im\alpha$, let $j\varepsilon$ be an arbitrarily chosen element of $j\alpha^{-1}$; if $j \in im\alpha$, let $j\varepsilon$ be an arbitrarily chosen element of [n]. Then it is clear that $i\alpha\varepsilon\alpha = i\alpha$ for all i in [n]. Of course may be a permutation, but be a permutation, but certainly $\eta = \varepsilon\alpha\varepsilon$ is singular, and $\alpha\eta\alpha = \alpha\varepsilon\alpha\varepsilon\alpha = \alpha\varepsilon\alpha = \alpha$.The semigroup $Sing_n$, has n-1 J-classes $J_1, ..., J_{n-1}$, where $J_r = \{\alpha \in Sing_n, |im\alpha| = r\}(r = 1, ..., n-1)$. Let E_{n-1} denote the set of idempotents in J_{n-1}. A typical element ε of E_{n-1} has image $[n]|\{i\}$ of cardinality n-1. The map ε acts identically on $[n]|\{i\}$, and sends i to some element $j \neq i$. We denote this map by $\begin{pmatrix} i \\ j \end{pmatrix}$; it maps i to j and all other elements identically. Notice that we can easily deduce that $E_{n-1} = n(n-1)$. □

Lemma 4.3.4. *Let $\alpha \in J_r$, where $1rn-1$. Then there exist ε in E_{n-1} and β in J_{r+1} such that $\alpha = \varepsilon\beta$.*

Proof. Write $im\alpha = (b_1, b_2, ..., b_r)$, and let $b_i\alpha^{-1}$= A_i, (i=1,2,...,r). It is convenient to write

$$\begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

in an obvious extension of a familiar notation. The sets A_i, form a partition of [n]. Since not all of the sets A_i, are singletons, we may assume without loss of generality that $A_1 = (a_1, a_1', ...)$ has at least two elements.

Then let $\varepsilon = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$,

$$\beta = \begin{pmatrix} A|\{a_1\} & A_2 & \dots & A_r|\{a_1\} \\ b_1 & b_2 & \dots & b_r b_{r+1} \end{pmatrix}$$

where $b_{r+1} \notin im\alpha$, and verify that $\alpha = \varepsilon\beta$. □

Corollary 4.3.5. *Every finite semigroup is embeddable in a finite semiband.*

Proof. Let S be a finite semigroup and let $X = S^1 \cup \{y, z\}$, where $y, z \notin S^1$. Define a map $\alpha : S \to T_X$ by $s\alpha = \rho s$, where
$x\rho_s = xs$ if $x \in S^1$,
$y\rho_s = z\rho_s = z$.. It is a routine matter to verify that α is a monomorphism. Moreover, it is clear that $s\alpha$ is a singular element of T_X for every s in S, and so α embeds S in the finite semiband $\mathrm{Sing}_{|x|}$. □

Theorem 4.3.6. *Every semigroup is embeddable in a semiband.*

Proof. Let S be a semigroup, and let T be a regular semigroup containing S. It is always possible to find such a semigroup T: for example, take $T = T_{S^1}$. Let I be a set containing a named element 1, and such that $|I/\{1\}|^2$ T, and define B to be the Rees matrix semigroup $\mathcal{M}[T^1; I; I; P]$, where the matrix $P = (p_{ij})$ over T^1 has the properties that $p_{i1} = p_{1i} = 1$ $(i \in I)$ and $T \subseteq \{p_{ij} : i, j \neq 1\}$.
The elements $(1, 1, i)$ and $(i, 1, 1)$ of B are evidently idempotent for all values of i. Also, since each t in T is equal to some pkl, we have $(i, t, j) = (i, 1, 1)(1, 1, k)(l, 1, 1)(1, 1, j)$, a product of idempotents. Thus B is generated by its idempotents. Next, B is regular, for if $(i, t, j) \in T$ and if t' is an inverse of t in the regular semigroup T, then
$(i, t, j)(1, t', 1)(i, t, j) = (i, tt't, j) = (i, t, j)$.
Finally, it is clear that the map $t \mapsto (1, t, 1)$ embeds T in B, and so S, as required, is embedded in a semiband B.

□

Conclusion

The regular semigroup which can be considered as the core semigroup since groups are regular semigroup with a unique idempotent. The idempotent plays a prodominat role in the structure of regular semigroup. Locally inverse semigroup and Orthodox semigroups are regular generalization of inverse semigroups. A regular semigroup S with set E of idempotents called locally inverse if eTe is an inverse. A Orthodox semigroup is regular semigroup in which the idempotent form a subsemigroup. The Band B is regular if it satisfies the identity. Completely regular semigroup form a prominent class of mathematical structures that have been extensively studied in semigroup theory, algebra and topology.

Reference

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