Syllabus for the First Degree Programme in Mathematics of the University of Kerala

Semester V
Real Analysis-I

CODE: MM 1541 — Instructional hours per week: 5
No.of credits: 4

The course on Real Analysis is spread over the last two semesters. Many of the topics discussed in this course were introduced somewhat informally in earlier courses, but in this course, the emphasis is on mathematical rigor.

In teaching this course, all ideas should be first motivated by geometrical considerations and then deduced algebraically from the axioms of real numbers as a complete ordered field. Also, the historical evolution of ideas, both in terms of physical necessity and mathematical unity should be discussed. (For a concise account of this, see en.wikipedia.org/wiki/Mathematical_analysis). Thus the course emphasizes the dialectic between practical utility and logical rigor on one hand and within mathematics, that between geometric intuition and algebraic formalism.

Module 1 The first step is to make precise the very concept of number and the rules for manipulating numbers. The course can start with a historical overview of how different kinds of numbers were constructed in different periods in history, depending on the physical or mathematical needs of the age. (See for example, http://en.wikipedia.org/wiki/Number#History and also [1]. The three articles on real numbers at
www-groups.dcs.st-and.ac.uk/~history/Indexes/Analysis.html
would also be of interest.)

A discussion on how real numbers are conceived as lengths and hence as points on a line should follow this. The approximation of irrational numbers by rational numbers, in the familar instances such as $\sqrt{2}$ and π, lead to an informal discussion of limits of sequences of rational numbers. This gives semi-rigorous definitions of operations on real numbers. The realization of the set $\mathbb{R}$ of real numbers as a field can be introduced at this stage and compared with the set $\mathbb{Q}$ of rational numbers, as in 2.1.1–2.1.4 of the textbook.

The idea of order in $\mathbb{Q}$ and $\mathbb{R}$ must be introduced next, as in 2.1.5–2.1.13 of the textbook. The notion of absolute value and that of a neighborhood, as in 2.2.1–2.2.9 of the textbook comes next.

Dedekind's translation of the "geometric continuity" of a line in the language of classes (see p.5 of [2]) can be discussed and phrased in modern set-theoretic terminology as the Completeness Axiom of $\mathbb{R}$ and can be used to *prove* the least upper bound property (see §§ 1.0–1.1 of [4] and also Eample 1.1 of [3]).

It should be emphasized at this point that the only assumptions we make about $\mathbb{R}$ are the axioms of a complete ordered field and every definition we make would be given in terms of these and every result we propose would be deduced from these axioms.

Applications of the completeness property, such as the Archimedean property (2.4.3 of the textbook) and its consequences (2.4.4–2.4.6), and also the density of rationals and irrationals (2.4.8 and 2.4.9) should also be discussed. Intervals in $\mathbb{R}$ deserve special mention. Their characterization and the nested interval property should be discussed as in 2.5.1–2.5.3 of the textbook.

Module 2 Having set the background, we now move on to the basic idea of mathematical analysis, that of limits. We first consider limits of sequences. The idea can be motivated by discussing the meaning of such equations as $\frac{1}{3} = 0.333\cdots$, $\sqrt{2} = 1.414213\cdots$ and $\pi = 3.1415\ldots$. The entire material in Chapter 3 of the textbook is to be discussed. The notion of absolute convergence, rearrangement, tests for absolute convergence as in section 9.2, excluding 9.2.6, should be discussed in connection with convegence of series.

Module 3 The idea of a cluster point of a set and the characterization of cluster points in terms of sequences, as discussed in 4.1.1–4.1.3 of the textbook must also be discussed here. The point that every rational number and every irrational number is a cluster point of the set of real numbers must be emphasized.

Limits of functions as in the rest of Chapter 4 of the textbook is also included in this part of the course. The historical evolution of the idea, as sketched in the introduction to this textbook, supplemented by relevant material from [1], should be first discussed. The rigorous $\epsilon - \delta$ definition of limits, to be discussed here, should be linked to the informal, mostly geometric, notion of limits used in earlier Calculus courses.

TEXT: Robert G Bartle: Introduction to Real Analysis, Third Ed., John Wiley & Sons

References

1. A. D. ALEXANDROV et al., *Mathematics:Its Content, Methods and Meaning*, Dover
2. R. DEDEKIND, *Essays on The Theory of Numbers*, available as a freely dowloadable e-book at `http://www.gutenberg.org/etext/21016`)
3. W. RUDIN, *Principles of Mathematical Analysis*, Second Edition, McGraw-Hill
4. A. E. TAYLOR, *General Theory of Functions and Integration*, Dover

DISTRIBUTION OF INSTRUCTIONAL HOURS:

Module 1: 30 hours; Module 2: 30 hours; Module 3: 30 hours